

# IDENTIFICATION OF SYSTEM PARAMETERS BY THE RANDOM DECREMENT TECHNIQUE

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## Abstract

The aim of this paper is to investigate and illustrate the possibilities of using correlation functions estimated by the Random Decrement Technique as a basis for parameter identification. A two-stage system identification method is used: first the correlation functions are estimated by the Random Decrement technique, and then the system parameters are identified from the correlation function estimates. Three different techniques are used in the parameter identification process: a simple non-parametric method, estimation of an Auto Regressive (AR) model by solving an overdetermined set of Yule-Walker equations and finally least square fitting of the theoretical correlation function. The results are compared to the results of fitting an Auto Regressive Moving Average (ARMA) model directly to the system output. All investigations are performed on the simulated output from a single degree-of-freedom system loaded by white noise.

## 1. Introduction

When loads acting on a system are measured together with the system responses, detailed information about system properties can be gained from the measurements. The problems of extracting information about physical properties from measurements are described in works about system identification, Ljung [1], Pandit and Wu [2], Söderström [3].

The system might be for instance an offshore structure loaded by sea waves and wind, and the responses might be accelerations in a number of selected points. In this case the system parameters describe structural properties like stiffness distribution, mass distribution and damping distribution. Using system identification for determination of physical properties of such structures can be advantageous for collection of general knowledge about loading conditions, for instance wave load spectra, and structural properties like the damping, but also for surveillance control of structural integrity. For instance, if a structural member is damaged, it will typically increase the damping and decrease the stiffness, and therefore structural damage can be estimated by system identification methods.

Let the physical properties be described by a limited number of scalars  $\theta_i$  forming the vector  $\underline{\theta} = \{\theta_i\}$  and let the values  $\underline{\theta}^*$ ,  $\hat{\underline{\theta}}$  be the true value and the value estimated by

system identification respectively. Since we are always dealing with limited information, the estimated properties  $\hat{\underline{\theta}}$  will not be identical to the true properties  $\underline{\theta}^*$ . However we want the deviation to be as small as possible. Therefore the method of system identification should be chosen in such a way that the systematic errors  $E[\hat{\underline{\theta}} - \underline{\theta}^*]$  - also denoted bias - and the elements of the covariance matrix  $Cov[\hat{\underline{\theta}} \hat{\underline{\theta}}^T]$  are minimum, Papoulos [4].

The time series formed by measuring loadings and responses of a system contains detailed information about the loadings and the system that cannot be fully represented by "interface functions" like spectral density estimates or correlation function estimates. This means that using interface functions usually will increase the covariance on the estimates, and if the interface functions are biased, bias will be introduced on the estimate of the system parameters. However, if an unbiased technique for estimation of the interface functions is used, the gain in speed may in some cases justify a small increase in the covariances. The speed is increased mainly because the interface functions are described by a small number of data points compared to the original time series and because the system properties can be obtained from the interface functions by simple techniques like linear regression.

There is only a limited number of techniques available for non-parametric estimation of interface functions. Fast Fourier Transform (FFT) spectral estimation is one possibility. However, such estimates will always be biased, Brigham [5]. Another possibility is to use correlation functions. In that case the interface functions can be obtained by several unbiased estimators: direct estimation (evaluating the correlation integral), the unbiased FFT correlation function estimator, Bendat and Piersol [7], Brincker et al [15] and finally different Random Decrement estimators, Brincker et al [15], Krenk and Brincker [14].

It has been shown however, Brincker et al [15], that the Random Decrement (RDD) technique for correlation function estimation is much faster than the direct method and the unbiased FFT. For short correlation function estimates for instance, the RDD technique is about 100 times faster than the unbiased FFT. Furthermore, the RDD estimates are even more accurate than the unbiased FFT for small damping. Therefore the RDD technique provides an excellent basis for a two-stage technique for identification of structural parameters.

In this paper a two-stage system identification technique based on using correlation functions estimated by RDD as interface functions is compared to fitting of Auto Regressive Moving Average (ARMA) models directly to the original time series. The auto correlation functions are estimated from the system output, and the system parameters are then identified from the auto correlation function estimates by three different techniques: a simple non-parametric technique, calibration of Auto Regressive (AR) models by solving the overdetermined set of Yule-Walker equations by linear regression and finally least square fitting of the theoretical correlation function by non-linear optimization.

The system output is simulated as the output from a single degree-of-freedom (SDOF) system loaded by white noise using an ARMA (2,1) model.

The accuracy of the techniques are compared estimating the variance and the bias of the estimated system parameters for different lengths of the correlation function, different sampling times and for different damping ratios of the system.

## 2. Simulation of System Response

In this paper autocorrelation functions are estimated for the response of a single degree-of-freedom (SDOF) system loaded by stationary Gaussian white noise. The response  $X(t)$  is the solution to the second order differential equation

$$\ddot{X} + 2\zeta\omega_0\dot{X} + \omega_0^2X = Q(t) \quad (1)$$

where  $\omega_0$  is the undamped natural angular frequency,  $\zeta$  is the damping ratio and  $Q(t)$  is stationary zero mean Gaussian white noise. For this case the normalized (corresponding to variance one) autocorrelation function for positive time lags is given by, Crandall and Mark [8]

$$R_{XX}(\tau) = \exp(-\zeta\omega_0\tau) (\cos(\omega_d\tau) + \frac{\zeta\omega_0}{\omega_d} \sin(\omega_d\tau)) \quad (2)$$

where  $\omega_d$  is the damped natural frequency  $\omega_d = \omega_0\sqrt{1-\zeta^2}$ . The most accurate way to perform simulations of a system formulated in continuous time, is to transform the system model to the discrete time space. This can be done by using an ARMA model. It can be shown, Pandit and Wu [2], that a second order system formulated in continuous time may be represented in the discrete time space by a (2,1) ARMA model given by

$$x_m = \Phi_1 x_{m-1} + \Phi_2 x_{m-2} + a_m - \Theta a_{m-1} \quad (3)$$

where  $m$  is the discrete time ( $t_m = m\Delta t$ ),  $\Phi_1$ ,  $\Phi_2$  are the Auto Regressive (AR) parameters,  $\Theta$  is the Moving Average (MA) parameter and  $a_m$  is a time series of independent

Gaussian distributed numbers with zero mean and variance  $\sigma_a^2$ . The model is denoted (2, 1) since it has 2 AR parameters and 1 MA parameter. If the ARMA parameters are chosen as

$$\Phi_1 = 2\exp(-\zeta\omega_0\Delta t) \cos(\omega_d\Delta t) \quad (4.a)$$

$$\Phi_2 = -\exp(-2\zeta\omega_0\Delta t) \quad (4.b)$$

$$\Theta = -P \pm \sqrt{P^2 - 1}; \quad |\Theta| < 1 \quad (4.c)$$

where

$$P = \frac{\omega_d \sinh(2u) - \zeta\omega_0 \sin(2v)}{2\zeta\omega_0 \sin(v) \cosh(u) - 2\omega_d \sinh(u) \cos(v)}$$

$u = \zeta_i\omega_i\Delta t$  and  $v = \omega_{di}\Delta t$ , then the ARMA model given by eq. (3) is the representation of the continuous system given in eq. (1) in the discrete time space. It can be shown, Pandit and Wu [2], that the discrete autocorrelation function of the time series  $x_m$  is equal to the sampled autocorrelation function of the continuous process  $X(t)$ . Therefore, the expectation of unbiased auto correlation function estimates obtained from the simulated time series will be equal to the theoretical auto correlation function, and a basis for unbiased estimation of system properties from simulations is established.

All simulations were performed using the AT version of the MATLAB software package, [17].

## 3. Estimation of Correlation Functions

Correlation functions are estimated using the Random Decrement (RDD) technique which is a fast technique for estimation of correlation functions for Gaussian processes by simple averaging. The RDD technique was developed at NASA in the late sixties and early seventies by Henry Cole and co-workers [9-12].

The basic idea of the technique is to estimate a co-called RDD signature. If the time series  $x(t)$ ,  $y(t)$  are given, then the RDD signature estimate  $\hat{D}_{XY}(\tau)$  is formed by averaging  $N$  segments of the time series  $x(t)$

$$\hat{D}_{XY}(\tau) = \frac{1}{N} \sum_{i=1}^N x(\tau + t_j) | C_{y(t_j)} \quad (5)$$

where the time series  $y(t)$  at the times  $t_j$  satisfies the trig condition  $C_{y(t_j)}$ , and  $N$  is the number of trig points. In eq. (5) a cross signature is estimated since the accumulated average calculation and the trig condition are applied to two different time series. If instead the trig condition is applied to the same time series as the data segments are taken from, an auto signature is estimated

It was shown by Vandiver et al, [13], who used the so-called level crossing trig condition  $C_{x(t)} : X(t) = a$  and

defined the RDD auto signature as the conditional expectation  $D_{XX}(t + \tau) = E[X(t + \tau) | X(t) = a]$ , that for the level crossing condition applied to a Gaussian process, the RDD signature is simply proportional to the auto correlation function

$$D_{XX}(\tau) = E[X(t + \tau) | X(t) = a] = \frac{R_{XX}(\tau)}{\sigma_X^2} a \quad (6)$$

where  $a$  is the trig level and  $\sigma_X^2$  is the variance of the process. This result has been generalized by Krenk and Brincker, [14], who showed that

$$\begin{aligned} D_{XY}(\tau) &= E[X(t + \tau) | Y(t) = a, \dot{Y}(t) = v] \\ &= \frac{R_{XY}(\tau)}{\sigma_Y^2} a - \frac{R'_{XY}(\tau)}{\sigma_Y^2} v \end{aligned} \quad (7)$$

where  $R'_{XY}(\tau)$  is the derivative of the correlation function and where  $\sigma_Y^2$  is the variance of the derivative process  $\dot{Y}(t)$ , for a Gaussian process given by  $\sigma_Y^2 = -R''_{YY}(0)$ . From this fundamental solution it is possible to explain the meaning of the RDD signature for several trig condition of practical interest

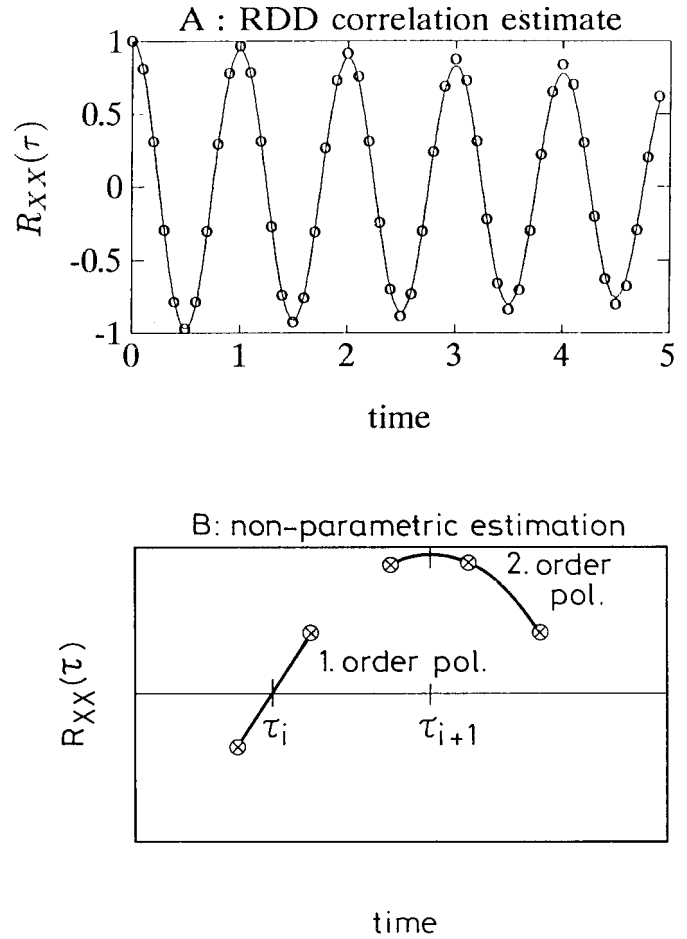
$$\begin{aligned} A: & Y(t) = a & \Rightarrow D_{XY}(\tau) \propto R_{XY}(\tau) \\ B: & \dot{Y}(t) = v & \Rightarrow D_{XY}(\tau) \propto R'_{XY}(\tau) \\ C: & \dot{Y}(t) = 0, Y(t) > a & \Rightarrow D_{XY}(\tau) \propto R_{XY}(\tau) \\ D: & Y(t) = 0, \dot{Y}(t) > v & \Rightarrow D_{XY}(\tau) \propto R'_{XY}(\tau) \end{aligned} \quad (8)$$

The result for trig condition A is found using that the distribution of  $\dot{Y}(t)$  for a Gaussian process is symmetrical and independent of  $Y(t)$ . The last term in eq. (7) will therefore vanish, and the result becomes proportional to the correlation function. The result for condition B is obtained by a similar argument, and the results for the conditions C and D follows directly from eq. (7).

The RDD estimates are "born" unbiased but the way the trig condition is implemented or the way trig points are selected from the time series might introduce either "window" bias or "trig point selection" bias, Brincker et al [16]. If, however the RDD auto signature is forced to be symmetric by taking the even part, and normalized to one for  $\tau = 0$ , then the signature will be an unbiased estimate of the auto correlation function for the output normalized to unit variance, Krenk og Brincker [14].

$$\hat{R}_{XX}(\tau) = \frac{D_{XX}(\tau) + D_{XX}(-\tau)}{2D_{XX}(0)} \quad (9)$$

In the following all correlation function estimates are obtained by using eq. (9) with the trig condition given by eq. (8c)  $\dot{X}(t) = 0, X(t) > a$ , where  $a$  is taken as  $a = \sigma_X$ .



**Figure 1.** 1a: a typical auto correlation function estimated by the Random Decrement technique for  $M = 50$ ,  $\Delta t = 0.1T$  and  $\zeta = 0.01$ .  $\circ$ : RDD estimate, solid line: exact solution. 1b: estimation of the extreme values (valleys and peaks) and crossing times on the auto correlation function and the crossing times of the derivative.

## 4. Identification of System Parameters

After the response has been simulated and auto correlation functions has been estimated by the RDD technique, the parameter estimate  $\hat{\theta} = [\hat{T} \ \hat{\zeta}]$  are obtained from the auto correlation estimates. A typical RDD auto correlation estimate is shown in figure 1a.

In this section three methods of identifying system properties from correlation function estimates are described: a simple non-parametric technique, estimation of AR models by linear regression and fitting of the theoretical auto correlation function by non-linear optimization. Finally it is described how system parameters can be estimated directly from the original time series by calibration of an ARMA model.

## A Simple Non-Parametric method

In the non-parametric method denoted RDD-NP the damping and the eigen period are found simply by estimating crossing times and the logarithmic decrement.

First all the extremes  $r_i$  - both peaks and valleys - on the correlation function are found. The logarithmic decrement  $\delta$  can then be expressed by the initial value  $r_0$  of the correlation function and the  $i$ 'th extreme

$$\delta = \frac{2}{i} \ln\left(\frac{r_0}{|r_i|}\right) \quad (10)$$

The logarithmic decrement and initial value of the correlation function can then be found by linear regression on  $i\delta$  and  $2\ln(|r_i|)$ , and the damping ratio are then. Thomson [6].

$$\zeta = \delta / \sqrt{\delta^2 + 4\pi^2} \quad (11)$$

A similar procedure was adopted for determination of the eigen period. The estimated times corresponding to the extremes and the zero crossing times of the correlation function forms an number of time points  $\tau_i$  equally spaced by a quarter of the damped eigen period. The damped eigen period  $T_d$  and the time shift (should be zero) of the correlation function can then be found by linear regression on the crossing times  $\tau_i$ , and the eigenperiod is then given by

$$T = T_d \sqrt{1 - \zeta^2} \quad (12)$$

The extreme values and the corresponding times were found by closed form expressions using 2. order polynomium interpolation, fig 1b. The crossing times of the correlation function itself were found by linear interpolation, figure 1b.

The extremes are a part of the envelope of the auto correlation function. The envelope only depends on the damping, and therefore even though the technique might seem simple, it provides a direct way of separating the problems of estimation of the damping and the eigen period. This observation is still true even when nonlinear damping is active, and therefore the simple non-parametric technique might be especially usefull in situations where the other techniques break down.

The simple non-parametric algorithm for estimation of the damping ratio and the eigen period of the auto correlation function was programmed in the C programming language and linked to the MATLAB environment by the MATLAB user function interface, [17].

## Auto Regressive (AR) Model Estimation

For a SDOF system loaded by white noise the stationary output in discrete time space is given by eq. (3). Using the white noise assumption  $E[a_i a_j] = \sigma_a^2 \delta_{ij}$  where  $\delta_{ij}$  is the Kronecker delta and the definition of the auto correlation function  $R_{XX}(\tau) = E[X(t + \tau)X(t)]$ , a difference equation for the autocorrelation function can be obtained by multiplying both sides of eq. (3) by  $x_{m-k}$  and then take the expectation. If the estimated auto correlation function is substituted for the theoretical auto correlation function, then the result becomes

$$\hat{R}_{XX}(t_m) = \Phi_1 \hat{R}_{XX}(t_{m-1}) + \Phi_2 \hat{R}_{XX}(t_{m-2}) \quad (13)$$

where  $t_m = m\Delta t$ ,  $1 < m < M$  and where  $M$  is the number of points in the estimated one-sided auto correlation function. This is a system of linear equations in the Auto Regressive (AR) Parameters  $\Phi_1$ ,  $\Phi_2$  often referred to as the Yule-Walker equations. When the number of equations are larger than the number of parameters to be estimated, the system becomes overdetermined, and the equations have no solution. For such situations however, standard methods exist for determination of approximate solutions. One possibility is to solve the system of equations by least square linear regression. In that case the estimate  $\hat{\Phi}_1$ ,  $\hat{\Phi}_2$  is called an overdetermined Yule-Walker estimate, Söderström [3].

From the estimates  $\hat{\Phi}_1$ ,  $\hat{\Phi}_2$  found by linear regression calibration of the AR model, the eigen period and damping ratio are found using the inverse closed form solutions given by eq. (4a) and (4b). The method of estimating the system parameters by AR calibration is denoted RRD-AR.

All the estimations were performed using the standard estimation function for the AR model in the MATLAB system identification toolbox, [17]

## Fitting the Theoretical Correlation Function

Since we know that we are dealing with a SDOF system, the form of the auto correlation function is known. An obvious way of estimating the system parameters  $\underline{\theta}$  is therefore to minimize the difference between the theoretical auto correlation function  $R_{XX}(\underline{\theta}, m\Delta t)$  given by eq. (2) and the estimated auto correlation function  $\hat{R}_{XX}(m\Delta t)$ . Using a least square approach, the method is denoted RDD-FIT and the parameter estimate  $\hat{\underline{\theta}} = [\hat{T} \ \hat{\zeta}]$  is found as the solution to the following set of nonlinear equations

$$\frac{\partial}{\partial \theta_i} \sum_{m=1}^M (R_{XX}(\underline{\theta}, m\Delta t) - \hat{R}_{XX}(m\Delta t))^2 = 0 \quad (14)$$

Typically the error function has many local minima, and therefore to prevent false solutions (not corresponding to the global minimum) a good initial estimate is essential. The nonlinear set of equations were solved using a quasi-Newton algorithm with a safeguarded mixed quadratic and cubic interpolation and extrapolation line search and using the parameter estimate from the non-parametric method as initial values.

All the estimations were performed using the standard optimization function FMINU in the MATLAB optimization toolbox, [17].

## Identification on the Original Time Series

From the simulation it is known that the true model is an ARMA (2,1). The ARMA models were calibrated using a predictor error method, Ljung, [1], and all estimations were performed using the standard calibration function in the MATLAB system identification toolbox.

## 5. Results

The accuracy of the techniques were investigated by identifying the system parameters for different sampling times  $\Delta T$ , different length's  $M$  of the auto correlation function estimate and for different damping ratios  $\zeta$ .

For each combination 20 time series of 4000 points each were simulated, the parameters were estimated from each of the time series as explained in the preceding sections, and the coefficient of variation  $\overline{std}[\hat{\theta}_i]/\theta_i^*$  and the empirical bias  $\overline{E}[\hat{\theta}_i - \theta_i^*]/\theta_i^*$  were calculated for the eigenperiod and the damping ratio, where  $\overline{std}[\ ]$  is the empirical standard deviation and  $\overline{E}[\ ]$  is the empirical expectation. The results are shown in figure 2, 3 and 4.

Figure 2 shows the identification results as a function of the sampling time  $\Delta t$  for  $\zeta = 0.001$  and  $M = 50$ . As it appears from the results the optimal sampling time is relatively close to the Nyquist frequency  $\Delta T = T/2$ . The most surprising results however seem to be the small differences between the four identification techniques. If the sampling time becomes to long then the RDD-NP technique becomes more unreliable than the other techniques. There is a tendency

for all the methods to over-estimate the damping, probably this is due to insufficient information in the relatively short time series.

Figure 3 shows the identification results as a function of the number of points  $M$  in the one-sided auto correlation function estimate for  $\Delta t = 0.4T$  and  $\zeta = 0.001$ . Again there is only a small difference in accuracy, although there is a tendency for the RDD-AR and ARMA estimates to be more accurate for short correlation function estimates. The RDD-AR estimates are very close to the ARMA estimates, figure 3a, 3b and 3d. For short auto correlation function estimates the NP estimates seem to be biased, and because of problems with formulating a flexible termination criterion for the nonlinear optimization this affects the RDD-FIT estimates through the initial values.

Figure 4 shows the results as a function of the damping ratio  $\zeta$ . As expected small damping ratios give a large coefficient of variation on the damping estimates, and a small coefficient of variation on the eigen period estimates. For small damping ratios all damping estimates seems to be biased. This is probably due to limited information about the damping in the time series - probably the time series becomes to short measured in correlation times. Again it appears from the results, that there is only a marginal difference between the accuracy of the four investigated techniques. For large damping ratios, the ARMA and the RDD-AR estimates are *systematic better than the RDD-NP and RDD-FIT estimates*, but for damping ratios in the structural range  $\zeta \in [0.01; 0.001]$  there seem to be no big difference.

Figure 5 shows the estimation times as a function of the length  $M$  of the correlation functions estimates. The dependency of  $M$  is small compared to mutual differences. In round figures the estimation times for the ARMA calibration was about 30 s, for the RDD-FIT technique about 1 - 2 s (10 - 20 iterations), for the AR estimation about 0.3 s and for the simple non-parametric technique about 10 - 20 ms.

The time for RDD estimation of the auto correlation function on a similar computer is about 0.1 s, Brincker et al [15]. This means that the total estimation times for the two-stage techniques are about 1-2 s for the RDD-FIT technique, about 0.4 s for RDD-AR estimation and about 0.1 s for the RDD-NP technique.

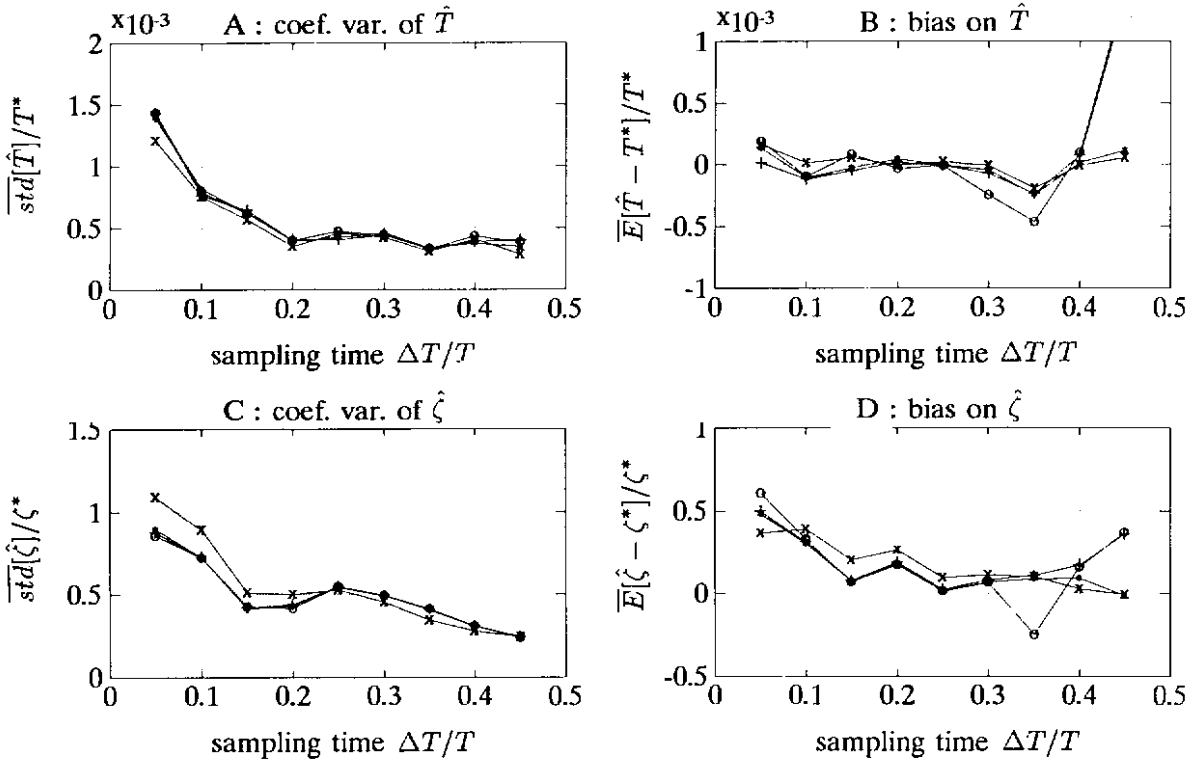


Figure 2. Coefficient of variation and bias on the estimated eigen period  $\hat{T}$  and the estimated damping ratio  $\hat{\zeta}$  as a function of the sampling time  $\Delta T$ . For all curves  $M = 50$ ,  $\zeta^* = 0.001$  and the symbols means:  $\circ$  : RDD-NP,  $*$  : RDD-AR,  $+$  : RDD-FIT,  $\times$  : ARMA.

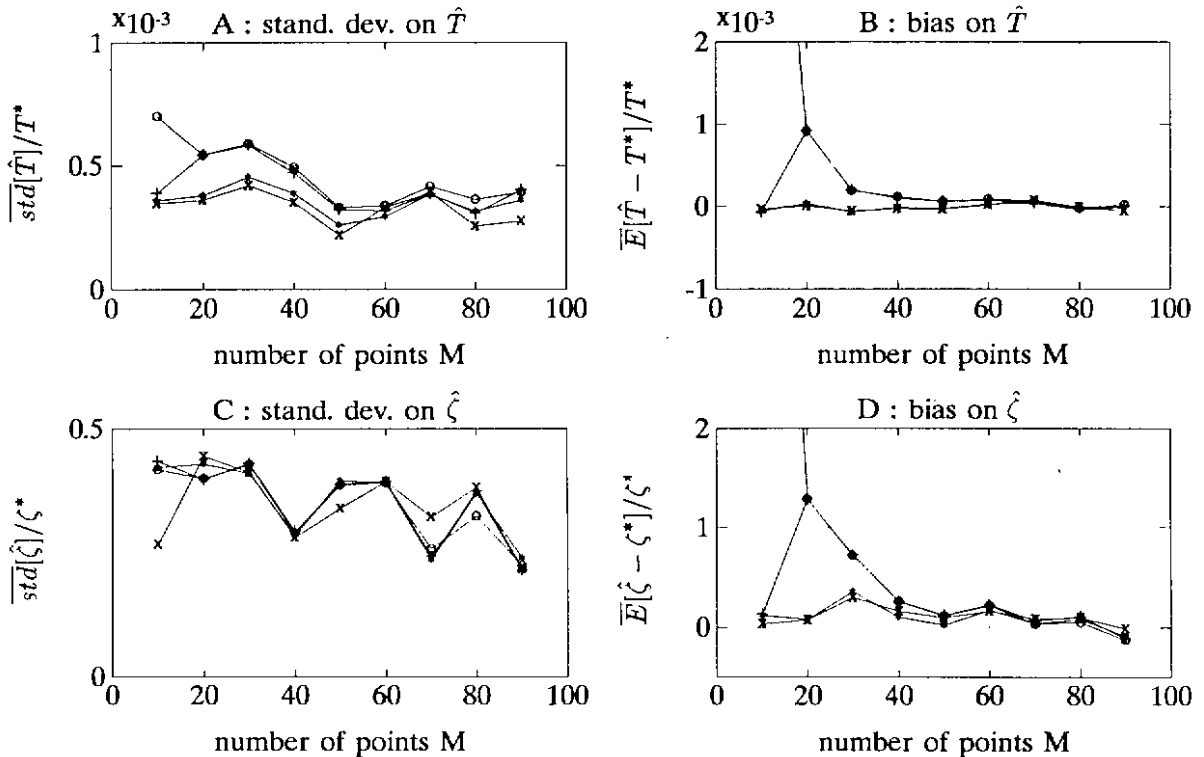


Figure 3. Coefficient of variation and bias on the estimated eigen period  $\hat{T}$  and the estimated damping ratio  $\hat{\zeta}$  as a function of the number of points  $M$ . For all curves  $\Delta T = 0.4T^*$ ,  $\zeta^* = 0.001$  and the symbols means:  $\circ$  : RDD-NP,  $*$  : RDD-AR,  $+$  : RDD-FIT,  $\times$  : ARMA.

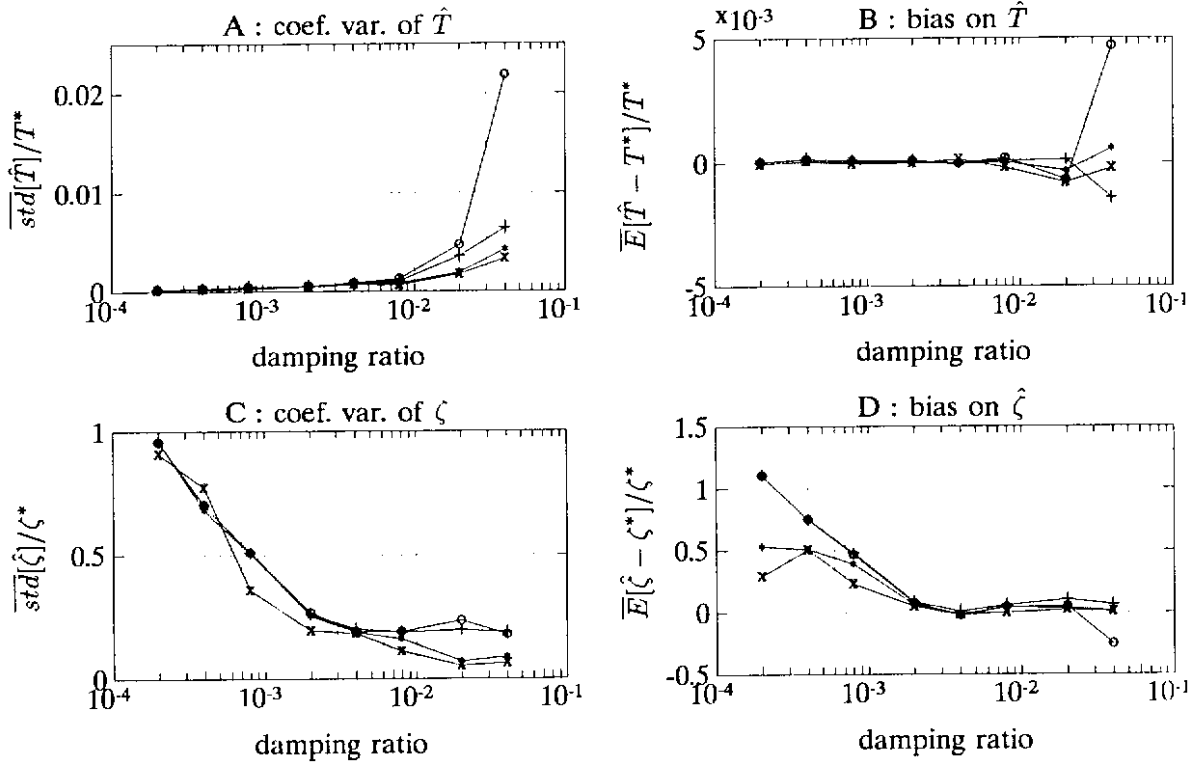


Figure 4. Coefficient of variation and bias on the estimated eigen period  $\hat{T}$  and the estimated damping ratio  $\hat{\zeta}$  as a function of the damping ratio  $\zeta$ . For all curves  $M = 50$ ,  $\Delta t = 0.4T$  and the symbols means:  $\circ$  : RDD-NP,  $*$  : RDD-AR,  $+$  : RDD-FIT,  $\times$  : ARMA.

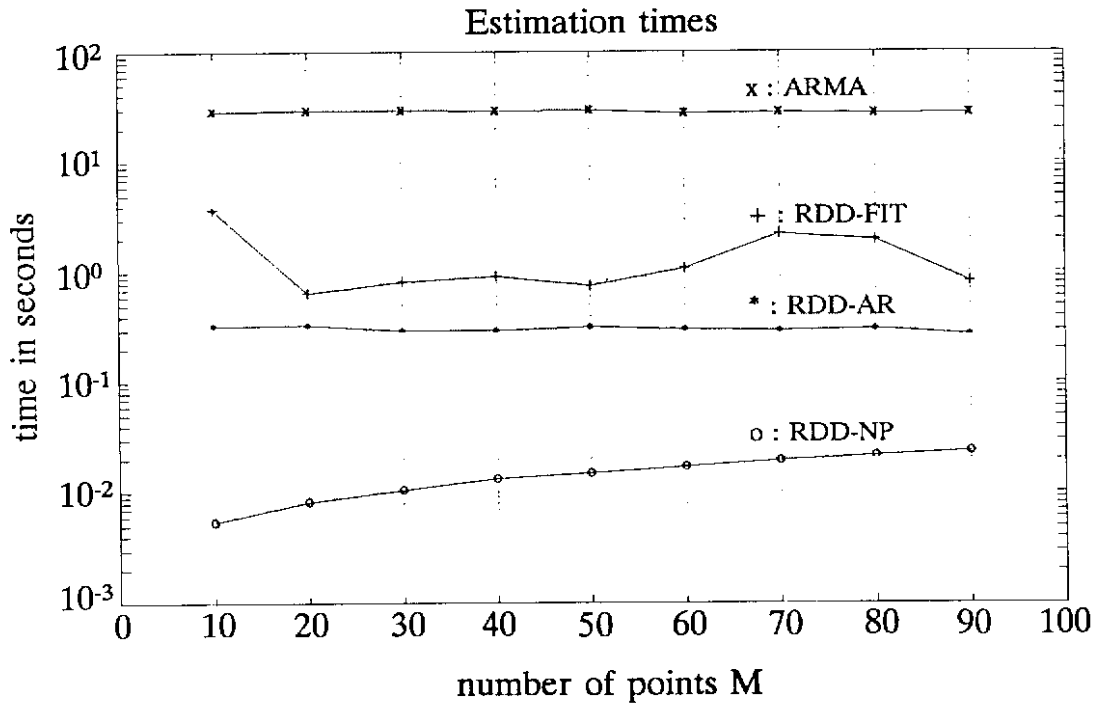


Figure 5. Estimation times for the different techniques as a function of the length  $M$  of the one-sided auto correlation function estimate using a 33 MHz 386-based PC with 387 co-processor.

## 6. Conclusions

On the basis of the results from the investigation of techniques for two-stage system parameter identification, the following conclusions can be drawn:

If a two-stage system identification technique is used where sufficiently long correlation function estimates are used as interface functions, then the use of interface functions only result in a relatively small loss of information. In this investigation about 50 points in the auto correlation function estimates was enough to ensure about the same estimation uncertainties as an effective one-stage technique (ARMA calibration).

If an unbiased correlation function estimator like the Random Decrement technique is used for interface function estimation, then the observed empirical bias on the parameter estimates are small compared to random errors in the most cases. When bias is observed, it seems to be introduced by limited information in the relatively short time series used in the investigation.

Three techniques was used for identification of system properties from the auto correlation function estimates: a simple non-parametric technique, calibration of Auto Regressive models and finally least square fitting of the theoretical auto correlation function. The three techniques did not differ very much in accuracy, although a detailed examination shows that AR estimation is the most accurate and the most reliable of the three techniques.

However the three techniques showed a significant difference in speed, the fit technique being slowest with about 1-2 s per estimation and the non-parametric being the fastest with about 10-20 ms per estimation.

The technique of fitting the theoretical auto correlation function cannot be recommended for future use. It was slow and unreliable compared to the other two techniques.

The simple non-parametric technique can be recommended. It is extremely fast and pretty reliable. Since the technique is based directly on the envelope of the correlation function, the technique might be especially usefull in cases with nonlinear damping where the parametric linear response models will break down.

Calibration of an AR model by solving the overdetermined set of Yule-Walker equations by least square linear regression seems to be a fast, accurate and reliable technique. It is nearly as accurate as calibration of an ARMA model directly on the original time series. It can be recommended for future applications.

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