Estimation of the Coefficient of Restitution of Rocking Systems by the Random Decrement Technique

Rune Brincker, Associate Professor of Structural Engineering
Aalborg University, Aalborg, Denmark

Milton Demosthenous, Research Assistant
George C. Manos, Professor of Earthquake Engineering
Aristotle University, Thessaloniki, Greece

Abstract

The aim of this paper is to investigate the possibility of estimating an average damping parameter for a rocking system due to impact, the so-called coefficient of restitution, from the random response, i.e. when the loads are random and unknown, and the response is measured. The objective is to obtain an estimate of the free rocking response from the measured random response using the Random Decrement (RDD) Technique, and then estimate the coefficient of restitution from this free response estimate. In the paper this approch is investigated by simulating the response of a single degree of freedom system loaded by white noise, estimating the coefficient of restitution as explained, and comparing the estimates with the value used in the simulations. Several estimates for the coefficient of restitution are considered, and reasonable results are achieved.

Nomenclature

t, τ : time
i, j, n : subscripts
$X(t), Y(t)$: response or stochastic process
$x(t), y(t)$: continous time series
h, b, r : ridig body geometry
μ : frictional coefficient
a, v : acceleration and velocity
g : gravity acceleration
e : coefficient of restitution
θ : rocking angle
θ_{cr} : critical rocking angle
M : body mass
I : body inertia moment
ν : natural rocking frequency
θ_0 : rocking amplitude
F : load
$s(\theta)$: sign function
e : coefficient of restitution
$D(\tau)$: RDD signature
$\hat{D}(\tau)$: RDD estimate
$C(t)$: trig condition at time t
N : number of trig points
$R_{YY}(\tau)$: auto correlation function for $Y(t)$
a, v : trig level, velocity
σ : standard deviation
$g(t), h(t)$: unit response, impulse response
$G(a, v, t)$: free response for initial conditions (a, v)
ΔT : time step
p_n : Gaussian distributed numbers
θ' : rocking angle just before impact
θ'' : rocking angle just after impact

1. Introduction

The dynamic response of solid-block structural systems excited by horizontal base motions might include sliding as well as rocking thus dissipating the kinetic energy in a different way from that of conventional structural systems.

To study the dissipation of energy in rocking systems, a simple one-degree-of-freedom system is considered, figure 1.a. The body and the base are assumed to be non-deformable, and in the initial state the ridig body is assumed to rest on the plane base having contact over the entire base area. Only simple contact exists between the body and the base without any other form of connection or attachment.

When the body is subjected to horizontal forces, either by applying external horizontal loads or by inertia forces arising from horizontal movements of the base, these forces will change the distribution of shearing stresses and normal stresses over the contact region. When the horizontal loads exceed a certain value, sliding or turning might occur.

Considering a rectangular body with a height h, a base width b and a body-base interface with simple frictional properties, the two modes are easily identified. If the horizontal and vertical acceleration of the base is a_h and a_v, respectively, then the body will slide if the horizontal acceleration is larger than the frictional resistance, i.e. if $a_h > \mu g(1 + a_v/g)$, where g is the gravity acceleration and μ is the coefficient of friction. Similarly it will rock if $a_h > (b/h)g(1 + a_v/g)$. Thus, slender bodies will have a tendency to rock, and compact bodies a tendency to slide, the transition being given by $\mu = b/h$.

However, for random base excitation such as earthquakes, because of the time varying characteristics of the excitation signal, the response will be a complicated combination of sliding and rocking, the mixture of the modes depending on the coefficient of friction μ and the slendernes ratio h/b.

If the body is slender, the sliding contribution will be small, and most of the kinetic energy is dissipated by the impacts occurring during rocking response. The dissipation of energy during impact is described by the so-called coefficient of restitution further defined in the following.

The objective of this paper is to investigate some possibilities for estimating the coefficient of restitution.

To obtain good estimates of physical parameters of mechanical systems, it is always an advantage to measure the response to a well-defined artificial loading. For instance, a simple and accurate way of estimating the coefficient of restitution is to make a free response test, [1]. For real structures, however, because of the risk of damage and the difficulties of exciting large structures in a well-defined way, this possiblity only exist in a few cases. Therefore, in practice, one might often accept to estimate the physical parameters from natural responses, in this case, from the response to earthqaukes.

Real earthquake excitation is not dealt with in this investigation. Instead, to simplify the problem only constant Gaussian white noise excitation is considered. Real earthquakes are more complicated. They are broadbanded but usually not close to white noise, and the transient phenomena are often of importance. It is believed however, that if the techniques developed here works well for the white noise case, the techniques can be revised and extended to be applicable also to earthquakes.

If both the excitation and the response of a rocking system are known, i.e. if accurate time series are awailable, then in principle it is possible to obtain good estimates of all the physical parameters of the rocking system by adjusting a mathematical model so that the difference between the measured response and the response of the model is minimum in some sense. This approch is often called System Identification, Ljung [2], and might be used for any kind of excitation, simple excitation like harmonic exciation or pulse excitation, white noise exciation or earthqauke excitation.

The system identification techniques are generally quite powerfull. However, they are often complicated and computationally expensive, and for non-linear systems the techniques are not fully developed. Further, the application of the technique assumes that a complete mathematical model can be formulated covering the behaviour of the real system in any way that is relevant to the consided situation. Thus, it might be difficult to use this approch to non-linear systems like a rocking system.

Therefore it is of value to develop simple techniques to give reasonable estimates of some of the most important physical parameters of rocking systems.

One of the simplest ways to get an estimate of the coefficient of restitution of a rocking system when only the response is known, is to use the Random Decrement technique. Using this technique an estimate of the free response is obtained from the random response by a simple averaging algorithm. A computer routine that calculates this estimate from the time series of the rocking response can be written in any language using only a few lines of simple code.

Once an estimate of the free response is obtained, the coefficient of restitution can be obtained in different ways. It is expected that the free response estimates obtained by the RDD technique will be biased, i.e. include systematic errors. Therefore it is expected that some ways of estimating the coefficient of restitution are better than others. To illustrate the bias problems, and to get an idea of how to estimate the coefficient of restitution from the free response estimates, intensive simulations were carried out in the following way.

By using the analytical solution for the free rocking response, the rocking response to a discrete Gaussian white noise horizontal base excitation was accurately simulated for a given value of the coefficient of restitution, the e_{target} value. This value of the coefficient of restitution is considered to be the exact value that also represents the target value for the following effort of obtaining its estimate. Next, from the simulated response to white noise excitation, free response estimates where obtained by employing the Random Decrement (RDD) technique as already mentioned. From these free response estimates an approximate value of the the coefficient of restitution was obtained. Different ways of obtaining this approximate value were considered, and the results were compared with the original value e_{target} thus getting a degree of the inherent bias in every case.

The results show, that it does not seem possible to obtain unbiased estimates of the coefficient of restitution using the RDD technique. However, one of the measures, a very simple measure based on the time lag for the first crossing and the first valley in the free response estimate, seems to be able to give estimates with an acceptable bias.

2. Rocking Response of a Single Solid Body

The rocking response of ridig bodies has been studied from the beginning of this century. A review of this research is given by Ishiyama [3]. Here only a short a presentation of the most important concepts for rocking of rigid bodies will be given

A rectangular body is considered, figure 1.a. It is assumed that the movement of the body will be pure rocking, i.e. pure angular displacement around one of the two bottom corners. If the rocking angle is smaller than the critical angle θ_{cr} (the overturning angle), se figure 1.a, then the moment will act as a restoring moment equal to $Mgr\sin(\theta_{cr} - \theta)$, where M is the mass of the body, g is the gravity acceleration, and r is the distance from the mass mid point of the body to one of the bottom corners. Using the equation of impulse momentum, it is easy to see, that if the body is loaded by the ground motion $\ddot{X}$, then the equations of motion are given by, Spanos et al [4]

$$
\begin{aligned}
I\ddot{\theta} + Mr\ddot{X}\cos(\theta_{cr}-\theta) + Mgr\sin(\theta_{cr}-\theta) &= 0\;;\quad \theta > 0 \\
I\ddot{\theta} + Mr\ddot{X}\cos(\theta_{cr}-\theta) - Mgr\sin(\theta_{cr}-\theta) &= 0\;;\quad \theta < 0
\end{aligned} \tag{1}
$$

where I is the inertia moment of the body around one of the bottom corners. As it appears from the equations of motion, the problem is non-linenar. The non-linearities are due to the term $\sin(\theta_{cr} - \theta)$, but also due to the change of sign when θ crosses zero. A complete set of boundary conditions is given by specifying $(\theta, \dot{\theta}) = (a, v)$. If it is assumed that the ground motions are zero, $\ddot{X} = 0$, then the solution to the equations of motion is the free response $G(a, v, t)$. If we specify $(\theta, \dot{\theta}) = (\theta_0, 0)$, and assume that the argument of the sine function is small so that $\sin(\theta_{cr} - \theta) \cong \theta_{cr} - \theta$, then the equations can be solved analytically, Housner [5]

$$
\theta(t) \;=\; G(a,0,t) \;=\; \theta_{cr} - (\theta_{cr} - \theta_0)\cosh(\nu t)\;;\quad \theta > 0 \tag{2}
$$

where $\nu = \sqrt{Mgr/I}$. This solution can be used to obtain the duration T of one full rocking period with the amplitude equal to θ_0, Housner [5]

$$
T \;=\; \frac{4}{\nu}\cosh^{-1}\left(\frac{1}{1-\theta_0/\theta_{cr}}\right) \tag{3}
$$

However, the simplest possible form of the equations of motion is obtained for small relative rocking angles, i.e. in the limit $(\theta/\theta_{cr}) \rightarrow 0$ and for slender systems (small θ_{cr}) where $\cos(\theta_{cr} - \theta) = 1$ and $\sin(\theta_{cr} - \theta) = \theta_{cr}$. In this case the equations of motion reduce to

$$
\ddot{\theta} \;=\; \nu^2\theta_{cr}s(\theta) + F \tag{4}
$$

where $s = -1$ for positive rocking angles, $s = 1$ for negative rocking angles and where F is the loading term $F = -Mr\ddot{X}$. For this equation, the free response consists of second order polynomials in time, and the rocking period is found as

$$
T \;=\; \frac{4}{\nu}\sqrt{2\theta_0/\theta_{cr}} \tag{5}
$$

The rocking periods given by eq. (3) and eq. (5) are shown in figure 1.b. As it appears from the figure, the rocking period T depends strongly on the amplitude θ_0. Further, if the relative rocking angle θ_0/θ_{cr} is smaller than approximately 0.1, then the differences between the solutions given by eq. (3) and eq. (5) are neglectable.

None of the above equations of motion include any damping term. It is usually assumed that energy is dissipated only by impact, i.e. when the rocking angle becomes zero, and the rotation point shifts from one side of the body to the other. Further it is assumed that, at every impact, the angular velocity is reduced by a constant factor, i.e. the angular velocity $\dot{\theta}''$ just after the impact is obtained from the angular velocity $\dot{\theta}'$ just before the impact by the simple reduction

$$
\dot{\theta}'' \;=\; e\,\dot{\theta}' \tag{6}
$$

where e is a constant - the so-called coefficient of restitution, usually a number a little below one. This assumption corresponds to assume that the kinetic energy is reduced by the factor e^2 at every impact. However, in every cycle there are two such impacts. Therefore after one cycle, the energy of the system is reduced by e^4, thus defining the constant logarithmic decrement δ, Thomson [6] given by

$$
\delta \;=\; \log(1/e^4) \tag{7}
$$

3. Estimation of Free Response

The Random Decrement (RDD) technique is a simple and fast technique for estimation of correlation functions for Gaussian pro cesses.

The Random Decrement (RDD) technique was developed at NASA in the late sixties and early seventies by Henry Cole and co-workers [7-10]. The basic idea of the technique is to estimate a so-called RDD signature by simple averaging. If the time series $y(t)$ is given, then the RDD signature estimate $\hat{D}(\tau)$ is formed by averaging N segments of the time series $y(t)$

$$
\hat{D}(\tau) \;=\; \frac{1}{N}\sum_{i=1}^{N} y(\tau + t_i)|\; C_{y(t_i)} \tag{8}
$$

where the time series $y(t)$ at the times t_i satisfies the trig condition $C_{y(t_i)}$, and N is the number of trig points. The trig condition might for instance be that $y(t_i) = a$ (the level crossing condition) or some similar condition.

Vandiver et al, [11] defined the RDD signature as the conditional expectation $D(\tau) = E[Y(\tau)|\; Y(0) = a]$, and proved that in the case of the level crossing trig condition applied to a zero mean Gaussian process, the RDD signature is simply proportional to the auto-correlation function R_{YY}

$$
D(\tau) \;=\; E[Y(\tau)|\; Y(0) = a] \;=\; \frac{R_{YY}(\tau)}{\sigma_Y^2}a \tag{9}
$$

where a is the trig level and σ_Y^2 is the variance of the process. This result was generalized to a more general trig condition by Brincker et al. [12-13]

$$
D(\tau) \;=\; E[Y(\tau)|\; Y(0) = a, \dot{Y}(0) = v] \;=\; \frac{R_{YY}(\tau)}{\sigma_Y^2}a - \frac{R'_{YY}(\tau)}{\sigma_{\dot{Y}}^2}v \tag{10}
$$

where $R'_{YY}(\tau)$ is the derivative of the correlation function and where $\sigma_{\dot{Y}}^2$ is the variance of the derivative process $\dot{Y}(t)$, for a Gaussian process given by $\sigma_{\dot{Y}}^2 = -R''_{YY}(0)$.

Now let $Y(t)$ be the response of a linear system loaded by white noise. In that case it is known, Crandall [14], that the normalized correlation function $R_{YY}(\tau)/\sigma_Y$ is equal to the free response $g(\tau)$ for a unit displacement and that the normalized derivative of the correlation function $R'_{YY}(\tau)/\sigma_{\dot{Y}}$ is equal to the free response $h(\tau)$

for a unit impulse, giving

$$D(\tau) = a\, g(\tau) + v\, h(\tau) \quad (11)$$

thus, for white noise loading, the RDD signature is a linear combination of the system functions $g(\tau)$ and $h(\tau)$ expressing the free response for the initial condition $(\dot{Y}, Y) = (a, v)$. Thus, for a linear system, a correlation function estimate might be obtained by the RDD technique or by some other technique, for instance the Fast Fourier Transform technique, Bendat and Piersol, [15], and if the system is loaded by white noise, the estimate can be considered as a free response.

One of the advantages with the RDD technique is, that in the case of a non-linear system, a similar result applies. This is not the case for Fast Fourier Transform estimates.

Now let the stochastic process $Y(t)$ be the zero mean rocking response from a single degree of freedom system. The free response for the initial conditions $H = [Y(0), \dot{Y}(0)] = [a, v]$ is given by $G(a, v, \tau)$, and we will consider the conditional response process

$$Y_c(t) = \{Y(t) \mid H = \eta\} \quad (12)$$

where $\eta = [a, v]$. Instead of conditioning on the response process $Y(t)$, we will condition on the load process $X(t)$. Thus, let us consider a load process $X_c(t)$, $t \le 0$ that is such that $H = \eta$. However, since it suffices only to consider the derivatives at $t = 0$, we can write

$$X_c(t) = \{X(t) \mid \Xi = \xi\} \quad (13)$$

where Ξ is a vector containing the derivatives $\Xi = [X(0), \dot{X}(0), \ldots]$ However, for any non-linear system the response is determined solely by the load history, $Y_c(t) = \mathcal{L}^t_{\tau=-\infty}\{X_c(\tau)\}$ where $\mathcal{L}^t_{\tau=-\infty}\{\}$ is a non-linear functional. Thus, since H is a complete set of initial conditions, then

$$Y_c(t) = G(a, v, t) \quad (14)$$

for cases where the load vanishes for positive times $X_c(t) \equiv 0$, $t > 0$. Therefore, if the load is non-vanishing for positive times, it is obvious to try to write the general solution in the form

$$Y_c(t) = G(a, v, t) + \mathcal{L}^t_{\tau=0+}\{X_c(\tau)\} \quad (15)$$

This is only true for systems where the principle of superposition is valid. In this case, for a non-linear system like a rocking system, this can only be assumed as an approximation. However, if we accept eq. (15) as an approximation, then the RDD signature, which is the expectation of the conditional process, is given by

$$\begin{aligned} D(\tau) &= E[Y_c(\tau)] \\ &= G(a, v, \tau) + E[\mathcal{L}^\tau_{t=0+}\{X_c(t)\}] \end{aligned} \quad (16)$$

The first part is the free response, and the second part is due to a finite length auto-correlation of the load process. The last part is in general not simple to evaluate. For this part, the expectation has to be carried out also for the multidimensional conditional variable Ξ, since many different load processes might result in the same H. However, if the load process is assumed to be white noise, the process has no memory, and therefore in this case conditions at $t = 0$ has no influence on the process for $t > 0$. The conditions can therefore be removed, and since the unconditional response is zero mean, the last term vanishes, and we get the simple result

$$D(\tau) = G(a, v, \tau) \quad (17)$$

Using this result to estimate the free response from the random response, it is not possible to consider all combinations of a and v. Furthermore, in practice one has to use finite size condition windows instead of excact conditions like $Y(0) = a$, $\dot{Y}(0) = v$. In practice, therefore, it is convenient to estimate the signatures

$$\begin{aligned} D_a(\tau) &= E[Y(\tau) \mid Y(0) = a, \dot{Y}(0) = 0] \\ D_v(\tau) &= E[Y(\tau) \mid Y(0) = 0, \dot{Y}(0) = v] \end{aligned} \quad (18)$$

by avaraging using the finite size window conditions

$$\begin{aligned} \hat{D}_a(\tau) &= \frac{1}{N_a} \sum_{i=1}^{N_a} y(\tau + t_i) \mid \;\; y(t_i) \in [a + \Delta a; a - \Delta a], \; \dot{y}(t_i) = 0 \\ \hat{D}_v(\tau) &= \frac{1}{N_v} \sum_{i=1}^{N_v} y(\tau + t_i) \mid \;\; y(t_i) = 0, \; \dot{y}(t_i) \in [v + \Delta v; v - \Delta v] \end{aligned} \quad (19)$$

For a broadbanded load process the relation between the RDD signatures and the free response is taken from eq. (17) as an approximation

$$\begin{aligned} D_a(\tau) &\cong g(a, \tau) \\ D_v(\tau) &\cong h(v, \tau) \end{aligned} \quad (20)$$

where $g(a, \tau) = G(a, 0, \tau)$, $h(v, \tau) = G(0, v, \tau)$. If the load process is not perfectly white noise, the RDD signatures will be coloured by the load process correlation function. However, in many cases the error might be reduced by removing unwanted frequences by filtering.

The asumption given by eq. (15) must be expected to cause bias errors in the free response estimates. However, since the free response estimates satisfy the proper initial conditions, the bias and its first derivative must be zero for time lag zero, thus, this bias will be small at the first part of the free response estimates.

The estimates $\hat{D}_a(\tau)$, $\hat{D}_v(\tau)$, however, also suffer from random errors due to the limited number of averagings and from bias introduced by the finite size windows $y(t_i) \in [a + \Delta a; a - \Delta a]$, and $\dot{y}(t_i) \in [v + \Delta v; v - \Delta v]$. In practice the choice of window sizes Δa, Δv is a trade off between bias and variance, i.e. small windows will give small bias but large variance, and vice versa. The bias can be removed by using different window sizes and extrapolating back to window size zero as shown later.

4. Response Simulation

The rocking response of a single degree of freedom system is simulated by using the simple version of the equation of motion given by eq. (4).

First a discrete white noise time series is obtained by simulating a series of N zero mean independent Gaussian distributed numbers p_n, $n = 1, 2, ..., N$. This time series is taken as the load term. However, since the time series is discrete, the values p_n has to be considered as impulses rather than load intensity.

Given the time series p_n, the equation of motion might be solved exactly using the following scheme.

The first value of the rocking angle is found from the initial conditions $(\theta_0, \dot{\theta}_0) = (a, v)$, and as long as the sign function $s(\theta)$ is constant, the next values $(\theta, \dot{\theta})$ are found from the recursive for mulas

$$\begin{aligned} \dot{\theta}_{n+1} &= \theta_n + \theta_{cr}\nu^2 s(\theta)\Delta T + p_{n+1} \\ \theta_{n+1} &= \theta_n + \dot{\theta}_n \Delta T + \frac{1}{2}\theta_{cr}\nu^2 s(\theta)\Delta T^2 \end{aligned} \quad (21)$$

where the time step ΔT is kept constant. In every step it is checked if the sign function $s(\theta)$ changes by obtaining the largest root τ from the equation

$$\theta_n + \dot{\theta}_n \tau + \frac{1}{2}\theta_{cr}\nu^2 s(\theta)\tau^2 = 0 \quad (22)$$

The largest root will always be positive. If $\tau > \Delta T$ the sign function remains constant during the next time step, and the equations (21) might be used. However, if $\tau < \Delta T$, the sign function will change during the next time step because θ will cross zero. Now let $(\theta_n^{'}, \dot{\theta}_n^{'})$ denote the rocking angle and the angle velocity just before crossing and let $(\theta_n^{''}, \dot{\theta}_n^{''})$ denote the corresponding quantities just after the crossing, then - in the case the sign function changes sign during the next time step - the next values of $(\theta, \dot{\theta})$ are found from

$$\begin{aligned} \dot{\theta}_{n+1} &= \dot{\theta}_n^{''} + \theta_{cr}\nu^2 s(\theta)(\Delta T - \tau) + p_{n+1} \\ \theta_{n+1} &= \theta_n^{''} + \dot{\theta}_n(\Delta T - \tau) + \frac{1}{2}\theta_{cr}\nu^2 s(\theta)(\Delta T - \tau)^2 \end{aligned} \quad (23)$$

where

$$\begin{aligned} \theta_n^{'} &= \theta_n^{''} = 0 \\ \dot{\theta}_n^{'} &= \dot{\theta}_n + \theta_{cr}\nu^2 s(\theta)\tau \; ; \quad \dot{\theta}_n^{''} = e\,\dot{\theta}_n^{'} \end{aligned} \quad (24)$$

and where of cause the sign function $s(\theta)$ changes sign at the crossing point. Figure 2 shows some examples of a simulated response for a coefficient of restitution of $e = 0.95$.

The solution scheme outlined above will give exact solutions, i.e. solutions that correspond exactly to the value of restitution supplied to the equations. In all simulations the value $e = 0.95$ was used.

The random responses Y were simulated with a resonably small time step so that a constant rocking signal with amplitude equal to the standard deviation of the response process would have approximately 75 simulated points per cycle.

RDD signatures were estimated from the signals using the idealized trig condition $(Y, \dot{Y}) = (a, 0)$ where the trig level a was taken as $a \cong 2\sigma_Y$, σ_Y being the standard deviation of the simulated response process. Two-sided RDD signatures were estimated, i.e. symmetrical averaging windows were used, and the free response estimate was obtained from the two-sided signature by taking only the even part of the signature in order to force the right velocity $(v = 0)$ condition on the estimate.

Figure 2.c shows a typical RDD free response estimate found as an average from 10 time series with 6000 points in each time series. As it appears from the figure, the RDD free response estimate fades out too fast, i.e. the estimate seems to be biased.

All simulations were performed using the MATLAB software package, [16].

5. Estimation of Coefficient of Restitution

In this paragraph different techniques for determination of the coefficient of restitution from the RDD free response estimates will be tested.

First we will consider a simple global measure and a simple local measure. From a statistical point of view, it is an advantage to use the whole free response estimate, therefore we will take a global measure of the coefficient of restitution by calculating the logarithmic decrement δ using all the extremes r_i of the free response estimate. The logarithmic decrement is expressed by the initial value r_0 of the free response estimate and the ith extreme r_i, Thomson [6]

$$\delta = \frac{2}{i}\ln(\frac{r_0}{|r_i|}) \quad (25)$$

and thus, the logarithmic decrement can be found by linear regression on $i\delta$ and $2\ln(|r_i|)$. The coefficient of restitution is then obtained from eq. (7). This estimate of the coefficient of restitution is denoted $\hat{e}_g$.

As explained above, the use of eq. (15) must be expected to impose bias on the free response estimate. However, the bias and its first derivative will be zero at time lag zero, i.e. the first part of the free response estimate must be expected to have much smaller bias than the tail. Also from the free responses shown in figure 2, it seems, that the bias is small on the first half cycle of the free response estimate and large on the tail. Therefore, it seems natural to estimate the coefficient of restitution from the first half cycle only. As a simple measure

$$\hat{e}_0 = \sqrt{\frac{|r_1|}{r_0}} \quad (26)$$

is taken where the initial value r_0 and the first minimum r_1 is obtained simply as the maximum value and the minimum value of the free response estimate.

However, the estimates also suffer from bias introduced by the finite size window Δa in eq. (19). This bias can be removed by estimating the coefficient of restitution for several window sizes and then extrapolate back to zero. The result of this analysis for $\hat{e}_g$ and $\hat{e}_0$ is shown in figure 3 showing the estimated values of the coefficient of restitution as a function of the relative window size $\Delta a/\Delta a_{max}$. The maximum window size Δa_{max} was taken as $\Delta a_{max} \cong 0.28a$ where a is the trig level. Each point in the figure is an estimate of the coefficient of restitution found from averaging 100 values each estimated from time series with 6000 points.

As it appears from Figure 3, the global measure, the estimate $\hat{e}_g$, is heavily biased as it would be expected. The value of $\hat{e}_g$ corresponding to zero window size was found as $\hat{e}_g = 0.8083$. The local measure using only the first part of the free response estimate is much closer to the exact value $e_{target} = 0.95$. The value of $\hat{e}_0$ corresponding to window size zero was found as $\hat{e}_0 = 0.9626$.

The extrapolation was made by linear regression. However, since the number of trig points varies with the window size, a large window will give small variance but large bias and vice versa. Therefore, the linear regression was carried out by minimizing the least square weighted by the variances, Hald [17]. Since the reciprocal variance will be approximately proportional to the number of trig points, and the number of trig points is approximately proportional to the window size, the least squares were simply weighted by the window size.

From this preliminary analysis, and from the theoretical considerations given above, it can concluded, that measures of the coefficient of restitution found by using a large part of the free response estimate will be heavily biased, and therefore not usuful for practical purposes. Further, it might be concluded, that local measures using only the first part of the free response estimate are not so heavily biased, and might be usefull for practical estimation. A detailed investigation was carried out to find the best way to estimate the coefficient of restitution from the first part of the free response estimate

In the detailed investigation four different methods were tested all using only the first half cycle of the free response estimate. Let the number of points in the first half cycle be N_h. Then the initial value c_1 was found fitting a second order polynomial to the $N_h/8$ first points of the estimate. Similarly, the first minimum c_2 and the corresponding time t_2 were found using $N_h/4$ points around the minimum. The slopes v_1 and v_2 just before and just after the crossing and the crossing time t_1 were found fitting a second order polynomial using $N_h/8$ points at each side of the crossing point. Since the ideal free response (for the simplified equations used in this paper) is known to be second order polynomials, four different measures of the the coefficient of restitution can easily be defined

$$\begin{aligned} \hat{e}_1 &= \sqrt{\frac{|c_2|}{c_1}} \\ \hat{e}_2 &= \frac{v_2}{v_1} \\ \hat{e}_3 &= \frac{|c_2|}{c_1}\,\frac{t_1}{t_2 - t_1} \\ \hat{e}_4 &= \frac{t_1}{t_2 - t_1} \end{aligned} \tag{27}$$

For each of these measures, the coefficient of restitution was obtained as a function of the window size as explained above. However, the simulation/estimation was repeated 20 times in order to investigate bias and random errors. The results are shown in Figure 4.

The empirical mean value and the empirical standard deviation corresponding to window size zero were found as

$$\begin{aligned} \overline{\hat{e}_1} &= 0.9628\,; & s_{\hat{e}_1} &= 0.0048 \\ \overline{\hat{e}_2} &= 0.9948\,; & s_{\hat{e}_1} &= 0.0055 \\ \overline{\hat{e}_3} &= 0.9717\,; & s_{\hat{e}_1} &= 0.0039 \\ \overline{\hat{e}_4} &= 0.9539\,; & s_{\hat{e}_1} &= 0.0065 \end{aligned} \tag{28}$$

It is easy to see, that the first three measures are biased, especially $\hat{e}_2$ is heavily biased. The last measure e_4 is the only measure that seems to be unbiased. However, a simple t-test show, that with a probability close to one, this measure is biased too. For this measure however, the bias is relatively small, and it seems resonable to accept this way of estimating the coefficient of restitution for practical purposes.

6. Conclusions

A technique has been developed and tested for estimation of the coefficient of restituion of rocking systems loaded by white noise. The technique is simple to apply, and only time series of the random rocking response has to be available.

The technique consists of two steps. First a free estimate is obtained by the Random Decrement Technique, and then from the free response estimate, the coefficient of restitution might be estimated. Because of non-ideal trig conditions and an approximate superposition assumption, it is to be expected that the Random Decrement free response estimates will be biased. However, the window bias might be removed by using several windows and extrapolating back to zero. Further, the influence of bias introduced by the superposition assumption might be minimized using only the very first part of the free response estimate.

Several ways of estimating the coefficient of restitution from the first part of the free response estimate were suggeested and intensive simulations were carried out to investigate bias and random errors of the suggested measures. Only one of the suggested measures showed an acceptably small bias. This measure is based only on the first crossing time of the free response estimate and the first crossing time of the derivative of the free response estimate.

The performed investigations were based on estimating the free response for the initial conditions $(Y, \dot{Y}) = (a, 0)$. However, since it can conluded that only the first part of the free response estimate is unbiased, it might be a better idea to obtain free response estimates for the initial conditions $(Y, \dot{Y}) = (0, v)$. Using the Random Decrement technique to obtain such an estimate, the (two-sided) derivative of the estimate will have a discontinuity at time lag zero directly corresponding to the coefficient of restitution, eq. (6), and therefore in this case the coefficient of restitution might be estimated using only a very narrow band around zero of the free response estimate.

This idea has to be investigated before final proposals can be made how to use the random decrement technique for estimation of the coefficient of restitution in practice. Also it has to be investigated if the technique can be used in case of non-white but broad-banded loading, and how the technique is to be used on real systems with several degrees of freedom.

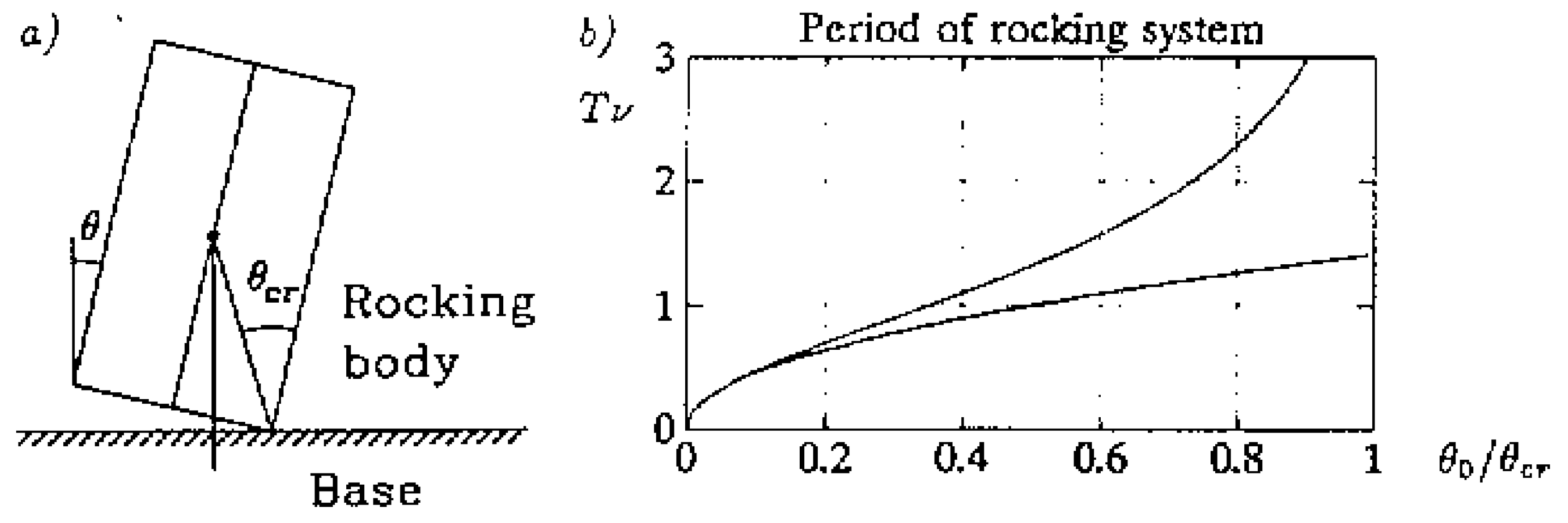


Figure 1. *Rocking system with one-degree-of-freedom*

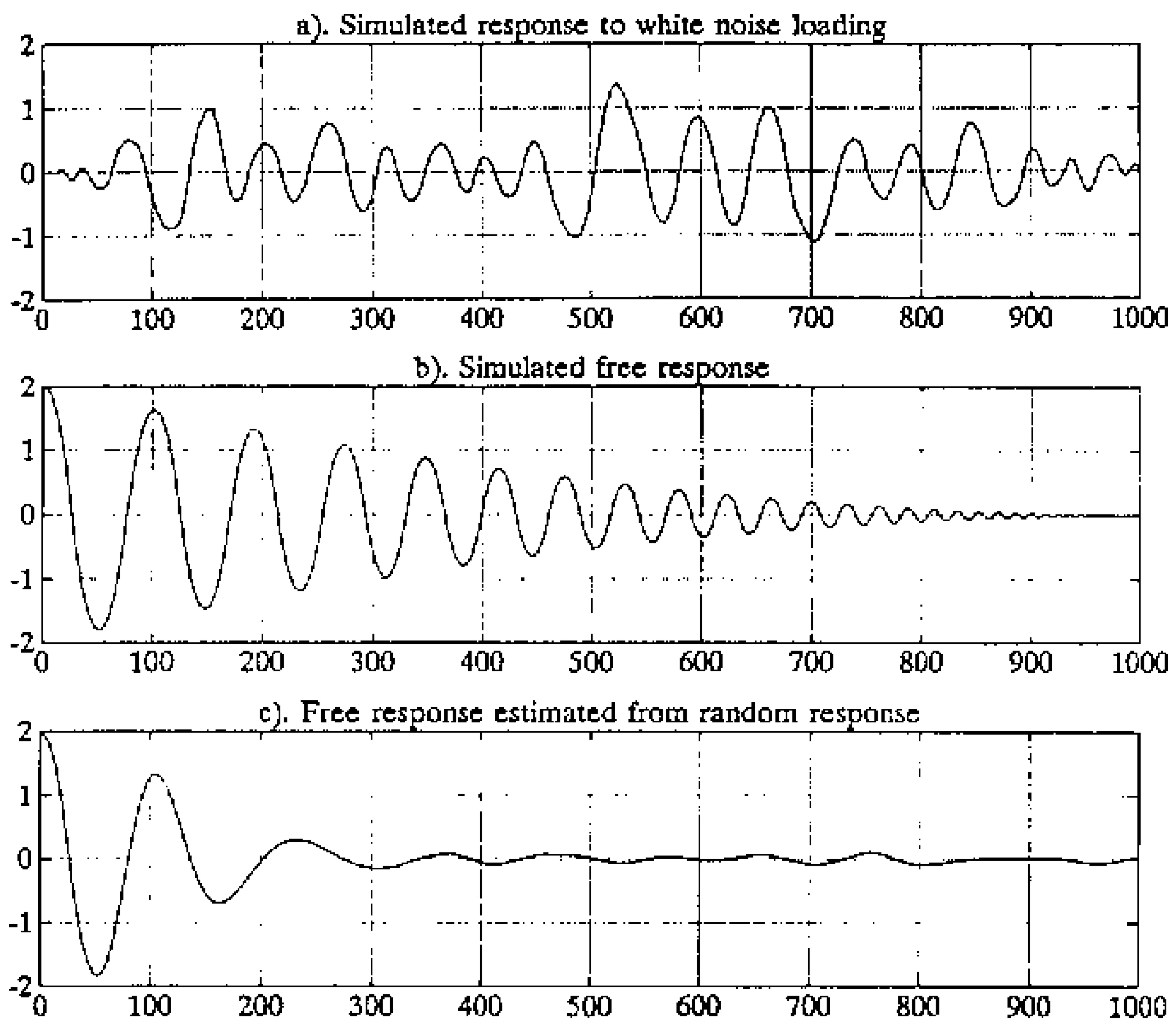
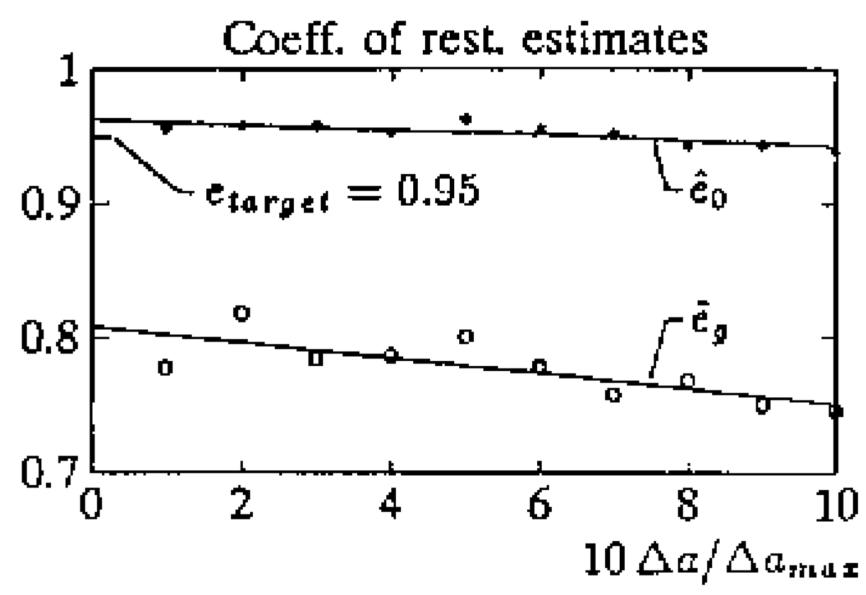


Figure 2. *Figures a) and b) show simulated responses for* $e = 0.95$. *Figure c) shows a free response estimate obtained from simulated random responses by the Random Decrement Technique.*

Figure 3. *The figure shows the two estimates* $\hat{e}_g$ *and* $\hat{e}_0$ *as a function of the relative window size* $\Delta a/\Delta a_{max}$.



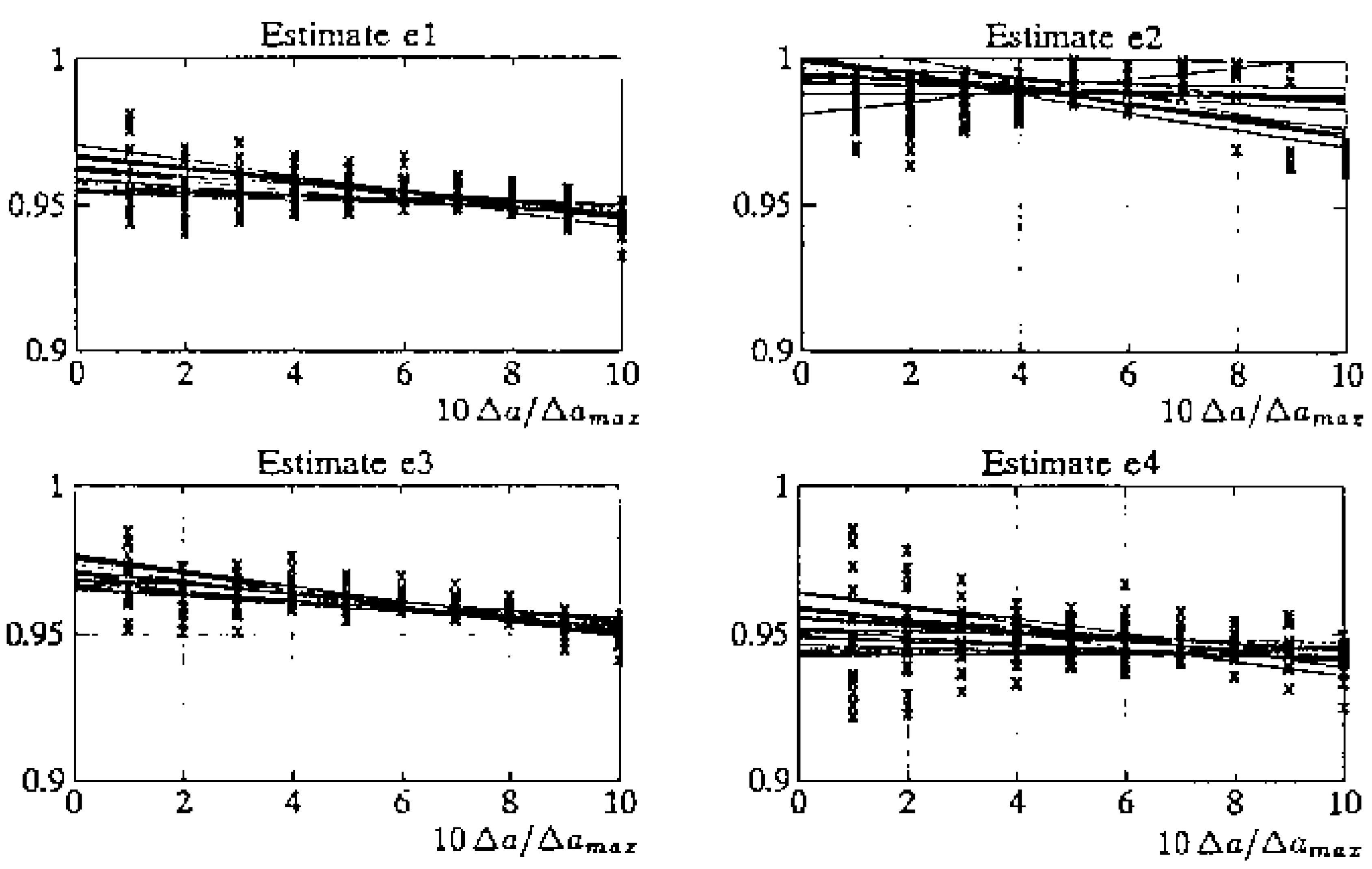


Figure 4. *20 repetitions of the four estimates* $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$, $\hat{e}_4$ *as functions of the relative window size* $\Delta a/\Delta a_{max}$, $e_{target} = 0.95$.

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