THEORY OF COVARIANCE EQUIVALENT ARMAV MODELS OF CIVIL ENGINEERING STRUCTURES

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ABSTRACT

In this paper the theoretical background for using covariance equivalent ARMAV models in modal analysis is discussed. It is shown how to obtain a covariance equivalent ARMA model for a univariate linear second order continuous-time system excited by Gaussian white noise. This result is generalized for multivariate systems to an ARMAV model. The covariance equivalent model structure is also considered when the number of channels are different from the number of degrees of freedom to be modelled. Finally, it is reviewed how to estimate an ARMAV model from sampled data.

NOMENCLATURE

$\boldsymbol{m}$	Diagonal mass matrix
$\boldsymbol{c}$	Symmetric damping matrix
$\boldsymbol{k}$	Symmetric stiffness matrix
T	Sampling period
$\boldsymbol{g}$	Modal weight of impulse response
$\boldsymbol{d}$	Modal weight of the covariance matrix
γ	Lagged covariance matrix of response process
σ_x^2	Covariance of a Gaussian white noise process $\boldsymbol{x}$
$\boldsymbol{y}(t)$	Continuous-time system response
$\boldsymbol{u}(t)$	Continuous-time Gaussian white noise
$\boldsymbol{x}(t)$	Continuous-time state vector
$\boldsymbol{h}(\tau)$	Impulse response function
$\boldsymbol{A}$	Continous-time state space matrix
$\boldsymbol{B}$	Continuous-time excitation matrix
$\boldsymbol{b}$	Continuous-time excitation vector
$\boldsymbol{m}_i$	Scaled modeshape
$\boldsymbol{M}$	Eigenvectors of $\boldsymbol{A}$
μ	Eigenvalues of $\boldsymbol{A}$
$\boldsymbol{\mu}$	Diagonal matrix of eigenvalues μ_i
$\boldsymbol{Y}_t$	Discrete-time system response
$\boldsymbol{a}_t$	Discrete-time Gaussian white noise
$\boldsymbol{X}_t$	Discrete-time state vector
$\boldsymbol{G}_k$	Green's function
$\boldsymbol{\phi}$	Discrete-time state space matrix
θ	Discrete-time excitation matrix
$\boldsymbol{L}$	Eigenvectors of $\boldsymbol{\phi}$
$\boldsymbol{l}_i$	Scaled modeshapes
λ	Eigenvalues of $\boldsymbol{\phi}$
$\boldsymbol{\lambda}$	Diagonal matrix of eigenvalues λ_i
$\boldsymbol{\phi}_i$	Auto-regressive polynomial coefficients
θ_i	Moving-average polynomial coefficients

1. INTRODUCTION

The use of non-parametric FFT-based methods has for many years been one of the most popular tools in modal analysis, but recently the interest in using parametrical models as the basis for modal analysis has increased. Since, the usual way of obtaining information about a structure is through sampling, all parametrical models are in some sense discrete equivalents to the continuous system. There are several methods for discretization. Some of these are approximation using pole-zero mapping, see Åström et al. [1], and approximation by hold equivalence techniques, see Safak [2]. But perhaps the most used approximation is the covariance equivalence technique, see Bartlett [3], Kozin et al. [4] and Pandit et al. [5]. In this paper the theoretical background for using covariance equivalent auto-regressive moving-average vector (ARMAV) models in modal analysis of civil engineering structures will be discussed. This is done by showing that a second order linear continuous-time system can be modelled by an ARMAV model. The results can immidiately be used as an effective simulation tool in case of white noise excitation, but can also be used for identification of structural systems from sampled data.

The theoretical considerations of this paper have been used in Kirkegaard et al. [6] for identification of the skirt piled Gullfaks C gravity platform, and in Brincker et al. [7] for identification of a multi-pile offshore platform.

In section two it will be shown how to obtain a covariance equivalent univariate (single channel) ARMA model of a single-degree of freedom system. Section three generalizes these results to a multivariate ARMAV model for a multi-degree of freedom system. The fourth section explains how to obtain a covariance equivalent univariate ARMA model for a multi-degree of freedom system. Finally, in the fifth section, it is described how these models can be calibrated to sampled data.

2. UNIVARIATE MODEL - SDOF SYSTEM

There are two criterions that must be satisfied in order to make an ARMA model covariance equivalent to an SDOF continuous-time linear system. Firstly, the modal properties must be equal, i.e. eigenfrequency and damping ratio must be the same. Secondly, the discrete-time autocovariance function of the system response must in some sense be equal to the continuous-time autocovariance function. The derivation therefore starts by considering a second order continuous-time system described by the differential equation

$$m\ddot{y}(t) + c\dot{y}(t) + ky(t) = u(t) \tag{1}$$

where m, c and k are the mass, damping and stiffness terms. $y(t)$ is the response of the system, and $u(t)$ is an independent distributed Gaussian white noise excitation with zero-mean and the variance σ_u^2. It is realized that the white noise approximation may not provide a very good approximation of non-white excitation, but it simplifies the autocovariance function and thereby also the resulting ARMA model. In section six, it will be explained how to deal with non-white excitation.

In continuous-time state space (1) is described by

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}u(t) \tag{2}$$

where $\boldsymbol{x}(t) = [y(t),\ \dot{y}(t)]^T$, and

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{c}{m} \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix} \tag{3}$$

The solution of (2) is given by

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \int_0^t e^{\boldsymbol{A}(t-s)}\boldsymbol{b}\,u(s)\,ds \tag{4}$$

and the response is the top half of $\boldsymbol{x}(t)$. In order to simplify (4) the continuous-time modal matrix $\boldsymbol{A}$ is decomposed

$$\begin{aligned} \boldsymbol{A} &= \boldsymbol{M}\boldsymbol{\mu}\boldsymbol{M}^{-1} \\ \boldsymbol{M} &= \begin{bmatrix} 1 & 1 \\ \mu_1 & \mu_2 \end{bmatrix} \\ \boldsymbol{\mu} &= diag\ \{\mu_i\}\ , \quad i = 1,2 \end{aligned} \tag{5}$$

where $\boldsymbol{\mu}$ is a diagonal matrix of distinct eigenvalues, and $\boldsymbol{M}$ is a Vandermonde matrix containing the corresponding eigenvectors. It is the eigenvectors that control the structure of $\boldsymbol{A}$, whereas the eigenvalues control the values of the bottom row of $\boldsymbol{A}$.

Inserting the decomposed modal matrix into (4) yields

$$\begin{aligned} \boldsymbol{x}(t) &= \boldsymbol{M}e^{\boldsymbol{\mu}t}\boldsymbol{M}^{-1} + \int_0^t \boldsymbol{M}e^{\boldsymbol{\mu}(t-s)}\boldsymbol{M}^{-1}\boldsymbol{b}\,u(s)\,ds \\ &= \boldsymbol{M}e^{\boldsymbol{\mu}t}\boldsymbol{M}^{-1} + \int_0^t \boldsymbol{h}(t-s)\,u(s)\,ds \end{aligned} \tag{6}$$

where $\boldsymbol{h}(\tau) = \boldsymbol{M}e^{\boldsymbol{\mu}\tau}\boldsymbol{M}^{-1}\boldsymbol{b}$ is the impulse response function of the state space system. The first part of (6) is the deterministic or transient part, and the last is the stochastic part.

From $\boldsymbol{h}(\tau)$ the impulse response of $y(t)$ can be extracted. Denoting this impulse response by $h^y(\tau)$ yields

$$\begin{aligned} h^y(\tau) &= \frac{1}{m(\mu_1 - \mu_2)}e^{\mu_1\tau} + \frac{1}{m(\mu_2 - \mu_1)}e^{\mu_2\tau} \\ &= g_{c1}e^{\mu_1\tau} + g_{c2}e^{\mu_2\tau} \end{aligned} \tag{7}$$

where the g_{ci}'s are modal weights. Assuming stationary conditions, i.e. initial values $\boldsymbol{x}(0) = \boldsymbol{0}$ and $|\mu| < 0$ for both eigenvalues, the autocovariance function of (1) is defined as

$$\begin{aligned} \gamma_c(\tau) &= \sigma_u^2 \int_0^\infty h^y(t)h^y(t+\tau)\,dt \\ &= d_{c1}e^{\mu_1\tau} + d_{c2}e^{\mu_2\tau}\ , \quad \tau \geq 0 \end{aligned} \tag{8}$$

where the d_{ci}'s also are modal weights, defined as

$$\begin{aligned} d_{c1} &= -\sigma_u^2\frac{g_{c1}^2}{2\mu_1} - \sigma_u^2\frac{g_{c1}g_{c2}}{\mu_1 + \mu_2} \\ d_{c2} &= -\sigma_u^2\frac{g_{c2}g_{c1}}{\mu_2 + \mu_1} - \sigma_u^2\frac{g_{c2}^2}{2\mu_2} \end{aligned} \tag{9}$$

Now turning to discrete time, it will now be shown that an ARMA(2,1) model with two auto-regressive parameters and one moving-average parameter is an adequate model. Defining the discrete response, Y_k as $y(kT)$, where T is the sampling period, the ARMA(2,1) model is given by

$$Y_t = -\phi_1 Y_{t-1} - \phi_2 Y_{t-2} + a_t + \theta_1 a_{t-1} \tag{10}$$

where ϕ_1, ϕ_2 are the auto-regressive parameters, and θ_1 is the moving-average parameter. a_t is an independent distributed Gaussian white noise with zero-mean and variance σ_a^2. Representing (10) in discrete-time state space yields

$$X_t = \phi X_{t-1} + \theta a_t \tag{11}$$

where $X_t = [Y_t,\ Y_{t-1}]^T$, $a_t = [a_t,\ a_{t-1}]^T$, and

$$\phi = \begin{bmatrix} -\phi_1 & -\phi_2 \\ 1 & 0 \end{bmatrix},\quad \theta = \begin{bmatrix} 1 & \theta_1 \\ 0 & 0 \end{bmatrix} \tag{12}$$

The solution of (11) is given by

$$X_t = \phi^t X_0 + \sum_{j=0}^{t} \phi^j \theta a_{t-j} \tag{13}$$

and the response Y_t is the top half of X_t. The discrete-time modal matrix, ϕ, can be decomposed in the following way

$$\begin{aligned} \phi &= L\lambda L^{-1} \\ L &= \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix},\quad \lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \end{aligned} \tag{14}$$

where λ is a diagonal matrix of distinct eigenvalues, and L is a Vandermonde matrix containing the corresponding eigenvectors. Again, it is the eigenvectors that control the structure of ϕ, and the eigenvalues that control the values of the top row of ϕ.

Inserting the decomposed modal matrix into (13) yields

$$X_t = L\lambda^t L^{-1} X_0 + \sum_{j=0}^{t} L\lambda^j L^{-1} \theta a_{t-j} \tag{15}$$

Similar to the continuous-time case the first part of (15) is the deterministic or transient part, and the last is the stochastic part.

In order to make the ARMA model covariance equivalent, the continuous-time system and the discrete-time system must be equal at all discrete time steps k, such that $t = kT$, for $k = 0, .., \infty$. The deterministic parts in (6) and (15) must therefore be equal. This is accomplished if $\lambda^k = e^{\mu kT}$, or equivalently if $\lambda = e^{\mu T}$ for all eigenvalues. The result that can be drawn from this is that a stable underdamped continuous-time SDOF system with two complex conjugated eigenvalues, also has two complex conjugated eigenvalues in discrete time. This is the reason for why it is necessary to have two auto-regressive parameters. So, at this point, the auto-regressive part of (10) can be constructed on the basis of the continuous-time modal matrix A in (3). Determine the eigenvalues, μ_i and the eigenvectors M. Calculate the discrete eigenvalues, λ_i, by using the identity $\lambda = e^{\mu T}$. Finally, calculate the discrete-time modal matrix, ϕ, using the similarity transformation in (14). The auto-regressive parameters will then be given by $\phi_1 = -\lambda_1 - \lambda_2$ and $\phi_2 = \lambda_1\lambda_2$.

However, the present auto-regressive model is not covariance equivalent, which is why it is necessary to add the moving-average. The reason for adding only one moving-average parameter can be seen by looking at the autocovariance function of (10). In order to calculate the autocovariance function, it is assumed that the initial values, X_0, and a_t, for $t < 0$, are zero. By applying the modal decomposition of (14) to (13), the response can be expressed in terms of the scalar Green's function, see Pandit [8] as

$$Y_t = \sum_{j=0}^{t} G_j a_{t-j} \tag{16}$$

where the Green's function is defined as

$$\begin{aligned} G_j &= \frac{\lambda_1 + \theta_1}{\lambda_1 - \lambda_2}\lambda_1^j + \frac{\lambda_2 + \theta_1}{\lambda_2 - \lambda_1}\lambda_2^j \\ &= g_1\lambda_1^j + g_2\lambda_2^j, \qquad j \geq 0 \\ &= 0, \qquad j < 0 \end{aligned} \tag{17}$$

and the g_i's are modal weights. Based on (16) and (17) the autocovariance function of (10), see Pandit et al. [5], is

$$\begin{aligned} \gamma_s &= \sigma_a^2 \sum_{j=0}^{\infty} G_j G_{j+s} \\ &= d_1\lambda_1^s + d_2\lambda_2^s\ , \quad s = 0,\ 1,\ 2,\ ... \end{aligned} \tag{18}$$

The modal weights in (18) are defined as

$$\begin{aligned} d_1 &= \sigma_a^2 \frac{g_1^2}{1-\lambda_1^2} + \sigma_a^2 \frac{g_1 g_2}{1-\lambda_1\lambda_2} \\ d_2 &= \sigma_a^2 \frac{g_2 g_1}{1-\lambda_2\lambda_1} + \sigma_a^2 \frac{g_2^2}{1-\lambda_2^2} \end{aligned} \tag{19}$$

To make the ARMA model covariance equivalent, the d_i's must be specified. This is done by using two initial conditions. However, one condition must always hold namely by the autocovariance at time lag zero, given by

$$\gamma_0 = d_1 + d_2 \tag{20}$$

Hence, only one initial condition remains to be specified, which is done by requiring one moving-average parameter. A method for calculation of θ_1 and σ_a^2 that easily conform to multivariate systems, is to express the autocovariance function implicitly using (10) and (17) as, see Pandit [8],

$$\gamma_k + \phi_1\gamma_{k-1} + \phi_2\gamma_{k-2} = \sigma_a^2 G_{-k} + \theta_1\sigma_a^2 G_{1-k}, \quad 0 \le k \le 1 \qquad (21)$$
$$= 0, \qquad k > 1$$

Using (21) for $k=1$ and $k=2$, and that $G_1 = \theta_1 - \phi_1$ gives the following second degree polynomial in θ_1

$$\theta_1^2 - \left(\frac{k_0}{k_1} + \phi_1\right)\theta_1 + 1 = 0 \qquad (22)$$

where

$$k_0 = \gamma_c(0) + \phi_1\gamma_c(T) + \phi_2\gamma_c(2T)$$
$$k_1 = \gamma_c(T) + \phi_1\gamma_c(0) + \phi_2\gamma_c(T) \qquad (23)$$

In (23) it is required that $\gamma_k = \gamma_c(kT)$, and at the same time used that $\gamma_k = \gamma_{-k}$. For each of the solutions of θ_1 in (22) corresponds a variance σ_a^2. This variance is determined by the following expression

$$\sigma_a^2 = \frac{k_1}{\theta_1} \qquad (24)$$

From (22) it is seen that there is in fact two covariance equivalent ARMA(2,1) models possessing the same modal properties.

3. MULTIVARIATE MODEL - MDOF SYSTEM

Now consider an MDOF continuous-time linear system. Such a system can be modelled by an ARMAV model. In order to make this model covariance equivalent, the same requirements as for the ARMA model must be imposed on it. In this section the procedure of the previous section will be generalized. Consider a system with n degrees of freedom, described by an $n \times n$ diagonal mass matrix $\boldsymbol{m}$, an $n \times n$ symmetric damping matrix $\boldsymbol{c}$, and an $n \times n$ symmetric stiffness matrix $\boldsymbol{k}$. It is assumed that the system is excited in all degrees of freedom by an independent distributed Gaussian white noise $\boldsymbol{u}(t)$ with covariance σ_u^2. Denoting the $n \times 1$ response vector by $\boldsymbol{y}(t)$, the continuous-time state space description of this system is

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \qquad (25)$$

where $\boldsymbol{x}(t) = [\boldsymbol{y}(t),\ \dot{\boldsymbol{y}}(t)]^T$, and

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{m}^{-1}\boldsymbol{k} & -\boldsymbol{m}^{-1}\boldsymbol{c} \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{m}^{-1} \end{bmatrix} \qquad (26)$$

In (26) $\boldsymbol{I}$ is an $n \times n$ identity matrix. The modal decomposition of $\boldsymbol{A}$ is given by

$$\boldsymbol{A} = \boldsymbol{M}\boldsymbol{\mu}\boldsymbol{M}^{-1}$$
$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{m}_1 & \cdots & \boldsymbol{m}_{2n} \\ \mu_1\boldsymbol{m}_1 & \cdots & \mu_{2n}\boldsymbol{m}_{2n} \end{bmatrix} \qquad (27)$$
$$\boldsymbol{\mu} = diag\ [\mu_i]\ , \quad i = 1,2,\ldots,2n$$

where $\boldsymbol{\mu}$ is a diagonal matrix of $2n$ distinct eigenvalues. $\boldsymbol{M}$ is a matrix containing the corresponding eigenvectors. The $\boldsymbol{m}_i$'s are scaled modeshapes. It is again the eigenvector and thereby the scaled modeshapes that control the structure of $\boldsymbol{A}$. The eigenvalues controls the values of the last n rows of $\boldsymbol{A}$.

Assuming zero initial conditions, the response $y(t)$ of (25) is

$$\boldsymbol{y}(t) = \int_0^\infty \boldsymbol{h}^y(t-s)\boldsymbol{u}(s)\,ds$$
$$\boldsymbol{h}^y(\tau) = \sum_{j=1}^{2n} \frac{\boldsymbol{m}_j\boldsymbol{m}_j^T}{S_j} e^{\mu_j\tau} = \sum_{j=1}^{2n} \boldsymbol{g}_{cj} e^{\mu_j\tau} \qquad (28)$$
$$S_j = \boldsymbol{m}_j^T\boldsymbol{c}\boldsymbol{m}_j + 2\mu_j\boldsymbol{m}_j^T\boldsymbol{m}\boldsymbol{m}_j$$

where $\boldsymbol{h}^y(t\text{-}s)$ is the $n \times n$ impulse response function, the matrices $\boldsymbol{g}_{cj}$ are modal weights, and S_j are the corresponding scalar modal masses, see Meirovitch [9]. Assuming stationary conditions the $n \times n$ lagged covariance function of the response is given by

$$\boldsymbol{\gamma}_c(\tau) = \int_0^\infty \boldsymbol{h}^y(\tau)\sigma_u^2\boldsymbol{h}^{yT}(t+\tau)\,dt$$
$$= \sum_{j=1}^{2n} \boldsymbol{d}_{cj} e^{\mu_j\tau} \qquad (29)$$

where the modal weights $\boldsymbol{d}_{cj}$ are defined as

$$\boldsymbol{d}_{cj} = -\sum_{i=1}^{2n} \frac{\boldsymbol{g}_i\sigma_u^2\boldsymbol{g}_j^T}{\mu_i + \mu_j} \qquad (30)$$

The resemblance between (8) and (29) is obvious. Turning to discrete time, it will now be shown that in the multivariate case, the ARMA(2,1) model expands to an ARMAV(2,1) model. Denoting $\boldsymbol{Y}_k$ as $\boldsymbol{y}(kT)$, the ARMAV(2,1) model is defined as

$$Y_t = -\phi_1 Y_{t-1} - \phi_2 Y_{t-2} + a_t + \theta_1 a_{t-1} \qquad (31)$$

where ϕ_1, ϕ_2 are the $n \times n$ auto-regressive matrices, and θ_1 is an $n \times n$ moving-average matrix. The $n \times 1$ vector a_t is an independent distributed Gaussian white noise with zero-mean and covariance matrix σ_a^2. Representing (31) in discrete-time state space yields

$$X_t = \phi X_{t-1} + \theta a_t \qquad (32)$$

where $X_t = [Y_t,\ Y_{t-1}]^T$, $a_t = [a_t,\ a_{t-1}]^T$, and

$$\phi = \begin{bmatrix} -\phi_1 & -\phi_2 \\ I & 0 \end{bmatrix},\quad \theta = \begin{bmatrix} I & \theta_1 \\ 0 & 0 \end{bmatrix} \qquad (33)$$

The modal decomposition of ϕ yields

$$\begin{aligned} \phi &= L\lambda L^{-1} \\ L &= \begin{bmatrix} \lambda_1 l_1 & \dots & \lambda_{2n} l_{2n} \\ l_1 & \dots & l_{2n} \end{bmatrix} \\ \lambda &= diag\ [\lambda_i]\ ,\quad i = 1,2,\dots,2n \end{aligned} \qquad (34)$$

where λ is a diagonal matrix of $2n$ distinct eigenvalues. L is a matrix containing the corresponding eigenvectors, and the l_i's are the scaled modeshapes. As for the continuous-time case, the only task for the eigenvectors and thereby the scaled modeshapes are to control the structure of ϕ. If the first n rows of ϕ were interchanged with the last n rows the structure of ϕ would be similar to the structure of A, i.e. the auto-regressive matrices would be in the n last rows and flipped in left and right direction. Because this is so, it can be verified that the scaled modeshapes of ϕ and A are equivalent.

As in the scalar case it is possible at this stage to determine the auto-regressive part of (31) on the basis of the continuous modal matrix A, defined in (26). The calculations follow the scalar case. Convert eigenvalues using the identity $\lambda = e^{\mu T}$. Calculate ϕ using a similarity transformation, keeping in mind that the scaled modeshapes m_i and l_i are equivalent.

The lagged covariance matrix γ_s of (31) can be expressed using the $n \times n$ matrix Green's function G. At a given time step j, G_j is defined using $2n$ modal weights g_i as

$$\begin{aligned} G_j &= \sum_{i=1}^{2n} g_i \lambda_i^j, && j \geq 0 \\ &= 0, && j < 0 \\ g_i &= l_i L^i (I\lambda_i + \theta_1) \end{aligned} \qquad (35)$$

where L^i is the ith row of the left $2n \times p$ submatrix of L . By using (35) the lagged covariance matrix can be defined in a similar manner as in the scalar case as

$$\begin{aligned} \gamma_s &= \sum_{j=0}^{\infty} G_j \sigma_a^2 G_{j+s}^T \\ &= \sum_{j=1}^{2n} d_j \lambda_j^s\ ,\quad s = 0,\ 1,\ 2,\ \dots \\ d_j &= \sum_{i=1}^{2n} \frac{g_j \sigma_a^2 g_i^T}{1 - \lambda_j \lambda_i} \end{aligned} \qquad (36)$$

On the basis of (35) and (36) it can be shown again that covariance equivalence can be obtained using only one moving-average matrix θ_1. Following the approach in the scalar case for calculation of θ_1 and σ_a^2, the multivariate equivalent to (21) is given by

$$\begin{aligned} \gamma_k + \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2} &= \sigma_a^2 G_{-k}^T + \theta_1 \sigma_a^2 G_{1-k}^T, && 0 \leq k \leq 1 \\ &= 0, && k > 1 \end{aligned} \qquad (37)$$

which provides the following second degree matrix polynomial in θ_1

$$\theta_1^2 - \theta_1 (k_0 + k_1 \phi_1^T) k_1^{-T} + k_1 k_1^{-T} = 0 \qquad (38)$$

where

$$\begin{aligned} k_0 &= \gamma_c(0) + \phi_1 \gamma_c(T)^T + \phi_2 \gamma_c(2T)^T \\ k_1 &= \gamma_c(T) + \phi_1 \gamma_c(0) + \phi_2 \gamma_c(T)^T \end{aligned} \qquad (39)$$

In (39) it is required that $\gamma_k = \gamma_c(kT)$. Further it is used that $\gamma_k^T = \gamma_{-k}$, and that the lagged covariance matrix is real. The solutions of θ_1 in (38) can be found using matrix polynomial techniques. It can be shown that there exist $K(2n,n)$ solutions. For each of these solutions correspond a covariance matrix σ_a^2, which is determined by

$$\sigma_a^2 = \theta_1^{-1} k_1 \qquad (40)$$

So again there are several covariance equivalent ARMAV models possesing the same modal properties.

4. UNIVARIATE MODEL - MDOF SYSTEM

In both cases considered until now the number of channels have been equal to the number of degrees of freedom. This approach has provided the maximum modal information, i.e. eigen-frequencies, damping and scaled modeshapes. In this section the covariance equivalence between a univariate ARMA model and a continuous-time multi-degree of freedom system, will be considered. The limitations of this approach are, that only eigenfrequencies and damping ratios can be determined.

If only one forcing function and output response for an MDOF system are considered, then (28) can be written as

$$y(t) = \int_0^\infty \boldsymbol{a}^T \boldsymbol{h}^y(t-s)\boldsymbol{b}\,u(s)\,ds \tag{41}$$

where $y(t)$ and $u(t)$ are scalars. The $n \times 1$ vector $\boldsymbol{b}$ is filled with zeroes except at the element that corresponds to the forcing function $u(t)$. The $n \times 1$ vector $\boldsymbol{a}$ is also filled with zeroes except the element that corresponds to the output response $y(t)$. Laplace transforming (41) yields

$$\begin{aligned} Y(z) &= H(z)U(z) \\ &= \sum_{j=1}^{2n} \frac{\boldsymbol{a}^T \boldsymbol{g}_{cj} \boldsymbol{b}}{z-\mu_j} \\ &= \frac{\sum_{j=1}^{2n} \boldsymbol{a}^T \boldsymbol{g}_{cj} \boldsymbol{b} \prod_{k=1,k\neq j}^{2n} (z-\mu_k)}{\prod_{i=1}^{2n}(z-\mu_j)} \end{aligned} \tag{42}$$

where $Y(z)$ and $U(z)$ are the Laplace transformed of $y(t)$ and $u(t)$, respectively, and z is any complex number. The last equation in (42) is in fact a scalar rational polynomial. The order of the denominator polynomial is $2n$, whereas the order of the numerator polynomial is $2n$-2. Applying the inverse Laplace transform to this rational polynomial yields a differential equation of $2n$ order of the following form

$$\begin{aligned} &(D^{2n} + \alpha_{2n-1}D^{2n-1} + \ldots + \alpha_0)y(t) = \\ &(D^{2n-2} + \beta_{2n-3}D^{2n-3} + \ldots + \beta_0)u(t) \end{aligned} \tag{43}$$

where D is a differential operator. The coefficients α_i and β_i can be calculated explicitly from the last equation in (42). The differential equation of (43) can be reduced to state space form by defining the state vector $\boldsymbol{x}(t) = [y(t)\ Dy(t)\ \ldots\ D^{2n-1}y(t)]^T$, and excitation vector $\boldsymbol{u}(t) = [u(t)\ Du(t)\ \ldots\ D^{2n-2}u(t)\ 0]^T$, and

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ . & . & \ldots & . \\ -\alpha_0 & -\alpha_2 & \ldots & -\alpha_{2n-1} \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \\ . & . & \ldots & . & . \\ \beta_0 & \beta_2 & \ldots & \beta_{2n-2} & 0 \end{bmatrix} \tag{44}$$

The resemblance with the SDOF state space formulation in (2) should be noted. By following the procedure used for univariate SDOF systems, the modal matrix $\boldsymbol{A}$ can easily be converted to a discrete modal matrix $\boldsymbol{\phi}$. This matrix will also be of the dimension $2n \times 2n$, i.e. corresponds to an ARMA model with $2n$ auto-regressive parameters.

The multivariate model (25) does not have derivatives of the forcing function on the right-hand side. Looking at the uni-variate model in (43), which is restricted to only one of the elements of the multivariate vector, it is seen that it does have derivatives of the forcing function on the right-hand side. Because the basic model is of second order, it can be verified that the derivative of the right-hand side never will exceed $2n$-2. Now, this result, of course, also holds for a univariate SDOF system, i.e. for n=1. In section two, it was shown that the covariance equivalent ARMA model of an SDOF system was an ARMA(2,1). So, by extending this result, the covariance equivalent model for an n-degrees of freedom univariate system is evidently an ARMA($2n$,$2n$-1).

The result is not restricted for multivariate to univariate models. Consider an n/m-variate system with n degrees of freedom. The covariance equivalent ARMAV model of such a system will then be an ARMAV($2m$,$2m$-1). As an ex-ample, consider a system with two channels and four degrees of freedom. This system can be modelled by an ARMAV(4,3) model.

6. IDENTIFICATION

In the previous sections it has been shown that it is possible to model any linear second order continuous-time structural system excited by Gaussian white noise using the ARMAV model. This result is usefull by itself, because it provides a very fast and easy simulation tool knowing $\boldsymbol{m}$, $\boldsymbol{c}$, $\boldsymbol{k}$, σ_u^2 and T. The result can on the other hand also be used to model a discretely sampled system. Consider a structural system without disturbance. Knowing that the sampled system contains n degrees of freedom in p channels the covariance equivalent ARMAV model is an ARMAV($2n/p$,$2n/p$-1). In the case of disturbance, e.g. non-white excitation of the structural system it may be necessary to increase the order of the model. By doing so, the non-white excitation is modelled as a part of the resulting model. The actual physical system can then be extracted from the model afterwards. This can be done using e.g. partial fraction expansion.

A well-known method for identification of a structural system is by applying the least square method to the ARMAV model. Consider a p-variate ARMAV(n,m) and a $p \times N$ matrix of samples y_t. A least-square criterion for such a model is typical of the following form, see Ljung [10]

$$
\begin{aligned}
V(\theta) &= \frac{1}{N}\frac{1}{2}\sum_{t=1}^{N} \varepsilon_t^T(\theta)\Lambda^{-1}\varepsilon_t(\theta) \\
\varepsilon_t(\theta) &= y_t - \hat{y}_t(\theta)
\end{aligned}
\tag{45}
$$

where $V(\theta)$ is the loss function, and ε_t is the prediction error. The $p \times p$ matrix Λ^{-1} weights together the relative importance of the components of ε_t. $\hat{y}_t(\theta)$ is the predictor of the model, defined as

$$
\begin{aligned}
\hat{y}_t(\theta) &= \phi_t^T\theta \\
\theta &= col(\phi_1\phi_2...\phi_n,\theta_1\theta_2...\theta_m) \\
\phi_t &= \varphi_t \otimes I_p
\end{aligned}
\tag{46}
$$

where θ is an $(n+m)p^2 \times 1$ parameter vector obtained by stacking all columns of the auto-regressive and moving-average matrices on top of each other.

The $(n+m)p^2 \times p$ regression matrix ϕ_t is obtained as the Kronecker product between a $p \times p$ identity matrix I_p, and the $(n+m)p \times 1$ regression vector φ_t, defined as

$$
\varphi_t = \begin{bmatrix} -y_{t-1} \\ . \\ -y_{t-n} \\ \varepsilon_{t-1} \\ \\ \varepsilon_{t-m} \end{bmatrix}
\tag{47}
$$

The parameters of the estimated model are the ones that minimize the loss function $V(\theta)$. In order to perform this minimization a numerical search procedure like the *Gauss-Newton* or the *Levenberg-Marquardt* is needed. In any case the numerical minimization will be non-linear because the prediction error ε_t depends on the estimated parameters. However, it is also possible to use linear multi-stage search procedure, see e.g. Piombo et al. [11].

7. CONCLUSION

In this paper the theoretical background for using covariance equivalent ARMAV models as a discrete equivalent of the linear second order continuous-time system, excited by Gaussian white noise, has been discussed. The correspondence between the number of channels of response, the number of degrees of freedom in the system, and the order of the ARMAV model has been shown. The results have shown that it is actually possible to model a continuous-time system explicitly in discrete time in a reasonable manner. It has also been considered how to identify structural system from sampled data using a non-linear least-square criterion.

8. ACKNOWLEDGEMENT

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