RANDOM DECREMENT BASED FRF ESTIMATION

R. Brincker & J.C. Asmussen

Department of Building Technology and Structural Engineering
Aalborg University, Sohngaardsholmsvej 57, 9000 Aalborg, Denmark

Abstract *The problem of estimating frequency response functions and extracting modal parameters is the topic of this paper. A new method based on the Random Decrement technique combined with Fourier transformation and the traditional pure Fourier transformation based approach is compared with regard to speed and quality. The basis of the new method is the Fourier transformation of the Random Decrement functions which can be used to estimate the frequency response functions. The investigations are based on load and response measurements of a laboratory model of a 3 span bridge. By applying both methods to these measurements the estimation time of the frequency response functions can be compared. The modal parameters estimated by the methods are compared. It is expected that the Random Decrement technique is faster than the traditional method based on pure Fourier Transformations. This is due to the fact that the Random Decrement technique is based on a simple controlled averaging of time segments of the load and response processes. Furthermore, the Random Decrement technique is expected to produce reliable results. The Random Decrement technique will reduce leakage, since the Fourier transformation will be applied to the Random Decrement functions, which has a natural decay.*

Nomenclature

a, v	Triggering levels.
C	Covariance functions.
C'	Time derivative of C.
D	Random Decrement function.
$\hat{D}$	Estimate of Random Decrement function.
h, $\mathbf{h}$	Impulse response function/matrix.
H, $\mathbf{H}$	Frequency response function/matrix.
N	Number of triggering points.
t, τ	Time variable.
$T_{X(t)}$	Triggering condition on X(t).
$X(t), Y(t)$	Measurements, time series.
$\dot{X}(t), \dot{Y}(t)$	Time derivative of $X(t), Y(t)$.
$Z(\omega)$	Fourier transform of $D(\tau)$.
$\gamma(\omega)$	Coherence function.
σ	Standard deviation.

1 Introduction

Usually the Frequency Response Function (FRF) of a linear structure is estimated from pure Fast Fourier Transformations (FFT) of meausurements of the response and the driving force. The most accurate way of estimating the FRFs depends on where and what kind of noise is introduced in the measurement. So dependent on the noise several different estimators of the FRFs exist, see e.g. Bendat & Piersol [1], Fabunmi et al. [2], Yun et al. [3]. Another problem with FFT based FRF's is the introduction of leakage. The leakage error is usually minimized by a suitable choice of window function applied in the time domain before calculating the FFT.

This paper deals with the same problems from another point of view. The measured input and output of a linear system is averaged in the time domain by applying the Random Decrement (RDD) technique. The theory behind the RDD technique and the link between the RDD functions and covariance function of zero mean Gaussian processes is described in Vandiver et al. [4] and Brincker et al. [5]. The concept described in this paper is more general and not restricted to Gaussian processes. In section 3 it will be shown that the Fourier trans-

form of the RDD functions constitute a basis for estimating FRFs.

Several advantages are expected. Since only a single Fourier transformation is performed and the RDD function is obtained by a simple averaging process in the time domain, the RDD approach is expected to be faster than the traditional approach in most cases. Furthermore, the RDD functions will decay towards zero which should eliminate the leakage problem. If noise is added to the measurements it will be averaged out in the time domain instead of the frequency domain.

The results presented in this paper are based on the measurements of the input and output of a laboratory bridge model. Previously, a simulation study was carried out, see Asmussen et al. [6]. The results encouraged to further investigations based on real data.

2 The Random Decrement Technique

The auto, D_{XX}, and cross, D_{YX}, RDD functions are defined as the mean value of the stochastic processes X and Y on some condition of X

$$D_{XX}(\tau) = E[X(t+\tau)|T_{X(t)}] \quad (1)$$

$$D_{YX}(\tau) = E[Y(t+\tau)|T_{X(t)}] \quad (2)$$

In eq. (1) and eq. (2) $T_{X(t)}$ is the triggering condition. The general applied triggering condition, $T^G_{X(t)}$, is introduced as

$$T^G_{X(t)} = \{a_1 \leq X(t) < a_2, v_1 \leq \dot{X}(t) < v\} \quad (3)$$

An unbiased estimate of the RDD functions is obtained by calculating the empirical conditional mean of the realizations of X and Y.

$$\hat{D}_{XX}(\tau) = \frac{1}{N}\sum_{i=1}^{N} X(t_i+\tau|T_{X(t_i)}) \quad (4)$$

$$\hat{D}_{YX}(\tau) = \frac{1}{N}\sum_{i=1}^{N} Y(t_i+\tau|T_{X(t_i)}) \quad (5)$$

If X and Y are stationary zero mean Gaussian processes and the general applied triggering condition is used, a fundamental relationship between the RDD functions and the covariance functions and their time derivative exists, see Brincker et al. [5]

$$D_{XX}(\tau) = \frac{C_{XX}}{\sigma_X^2}a - \frac{C'_{XX}(\tau)}{\sigma_{\dot{X}}^2}v \quad (6)$$

$$D_{YX}(\tau) = \frac{C_{YX}}{\sigma_X^2}a - \frac{C'_{YX}(\tau)}{\sigma_{\dot{X}}^2}v \quad (7)$$

where the triggering levels a and v are determined from the density function of X and $\dot{X}$ and the triggering levels

$$a = \int_{a_1}^{a_2} x f_X(x)dx \qquad v = \int_{v_1}^{v_2} \dot{x} f_{\dot{X}}(\dot{x})d\dot{x} \quad (8)$$

Several different triggering conditions can be formulated in order to pick out only the covariance functions or their time derivative, Brincker et al. [5]. In this paper only the positive point triggering condition, $T^P_{X(t)}$, is used

$$T^P_{X(t)} = \{a_1 \leq X(t) < a_2\} \quad (9)$$

Since no condition are made on $\dot{X}(t)$ and the mean value is assumed to be zero, the triggering level v is zero according to eq. (8), eq. (6) and eq. (7) are reduced to

$$D_{XX}(\tau) = \frac{C_{XX}(\tau)}{\sigma_X^2}\cdot a \quad (10)$$

$$D_{YX}(\tau) = \frac{C_{YX}(\tau)}{\sigma_X^2}\cdot a \quad (11)$$

The estimation time of the RDD functions is dependent on three user options. The actual choice of triggering condition, the choice of triggering levels and the choice of the maximum time lag in the RDD functions. Eq. (6) and eq. (7) constitute the basis for using the RDD technique in ambient testing by assuming the unmeasurable load to be white noise or white noise filtered through a rational shaping filter. The modal parameters can then be extracted from the RDD functions using methods which are based on free decays or impulse response functions.

If the load and the reponse of a linear mechanical system are measured another approach for estimating modal parameters by the RDD technique exists. The Fourier transformation of the RDD functions can be used to estimate the FRFs of the system.

3 Estimation of FRFs

The response of a viscous damped linear mechanical system with n degrees of freedom is given by the convolution or Duhamel integral

$$\mathbf{Y}(t) = \int_{-\infty}^{t} \mathbf{h}(t-\eta)\mathbf{X}(\eta)d\eta \quad (12)$$

The response of the ith mass to any load applied at the jth mass is

$$Y_i(t) = \int_{-\infty}^{t} h_{ij}(t-\eta)X_j d\eta \quad (13)$$

Using substitution of variables ($t = t+\tau$, $\eta = \xi+t$) eq. (13) is rewritten to

$$Y_i(t+\tau) = \int_{-\infty}^{\tau} h_{ij}(\tau-\xi)X_j(t+\xi)d\xi \quad (14)$$

The impulse response function is assumed to be time invariant. Applying the definition of the RDD function reduces eq. (14) to

$$D_{Y_iY_i}(\tau) = \int_{-\infty}^{\tau} h_{ij}(\tau-\xi)D_{X_jY_i}(\xi)d\xi \quad (15)$$

Or alternatively

$$D_{Y_iX_j}(\tau) = \int_{-\infty}^{\tau} h_{ij}(\tau-\xi)D_{X_jX_j}(\xi)d\xi \quad (16)$$

The RDD functions in eq. (15) and (16) are not dependent on any particular formulation of the triggering condition. The Fourier transformation, $Z(\omega)$ of the RDD function $D(\tau)$ is defined as

$$Z(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega\tau}D(\tau)d\tau \quad (17)$$

The Fourier transformation of both sides of eq. (15) and eq. (16) is given by

$$Z_{Y_iY_i}(\omega) = H_{ij}(\omega)Z_{X_jY_i} \quad (18)$$

$$Z_{Y_iX_j}(\omega) = H_{ij}(\omega)Z_{X_jX_j} \quad (19)$$

Corresponding to pure FFT analysis a coherence function based on the Fourier transformed RDD functions can be calculated

$$\gamma(\omega) = \frac{|Z_{XY}(\omega)|^2}{Z_{XX}(\omega)Z_{YY}(\omega)} \quad (20)$$

Eq. (18) and eq. (19) are both estimators of the FRFs. Which of these two estimators to use depends on the noise included in the measurements.

4 Choice of Estimator

Suppose that the measured response $Y_M(t)$ at some point on the structure consists of the true structural response, $Y(t)$ and measurement noise, $W(t)$. The measured input is assumed to be noise free

$$Y_M(t) = Y(t) + W(t) \quad X_M(t) = X(t) \quad (21)$$

In order to prevent introduction of false triggering points the RDD functions are calculated as

$$\begin{aligned} D_{X_MX_M}(\tau) &= E[X_M(t+\tau)|T^G_{X_M(t)}] \\ &= D_{XX}(\tau) \end{aligned} \quad (22)$$

$$\begin{aligned} D_{Y_MX_M} &= E[(Y(t+\tau)+W(t+\tau))|T_{X_M(t)}] \\ &= D_{YX}(\tau) + D_{WX}(\tau) \end{aligned} \quad (23)$$

If the noise, W, is independent of the input X and has zero mean, the last term vanishes. This means that in the case of independent noise at the response only the identification should be based on eq. (19). On the other hand, independent zero mean measurement noise at the input only has the effect that the identification should be based on eq. (18).

5 Laboratory Bridge Model

The laboratory bridge model consists of a simply supported steel plate with 3 spans. The steel plate has the dimensions 3.0 × 0.35 m. The length of each span is 1 m. A shaker is attached at the right-hand span. The shaker is exciting the bridge model with white noise in the frequency span 0-60 Hz. The measurements consist of 32000 points sampled with 150 Hz. The measurements are filtered analogously and digitally after sampling.

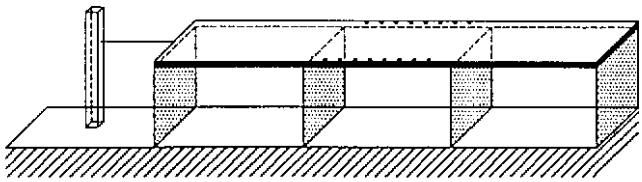

Figure 1: *Laboratory bridge modal and sensor locations.*

6 Results

The FRFs of the bridge model are estimated using traditional FFT and the method of combining RDD and FFT. The modal parameters are extracted from the IRFs by the polyreference time domain technique, see Vold et al. [7]. The physical modes are extracted from the computational modes using a criterion on the damping ratios and modal confidence factors, see Vold et al. [8].

Figure 2 and figure 3 show typical FRFs obtained from pure FFT and RDD-FFT applied to the same data set.

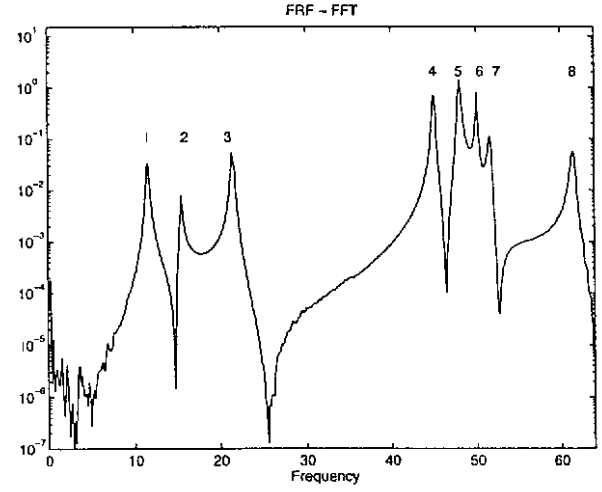

Figure 2: *Typical FRF, $|H(\omega)|^2$, estimated using pure FFT. The mode number is indicated.*

Two differences are seen at the FRFs. First the RDD-FFT based FRF seems to be influenced by more noise compared to the pure FFT based FRF. Second the peaks at the RDD-FFT based FRF looks shaper or they include less damping than the FFT based FRF

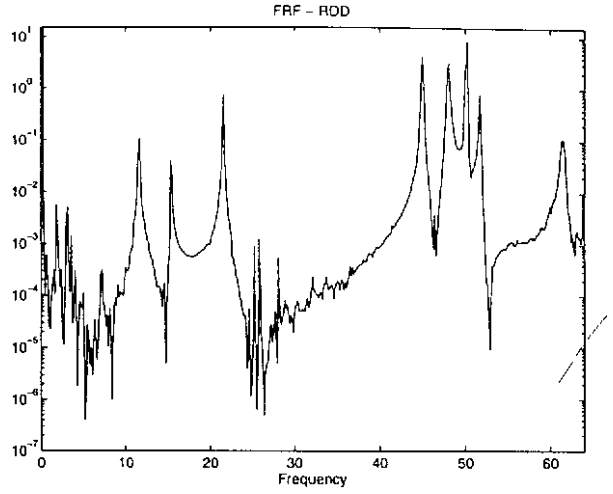

Figure 3: *Typical FRF, $|H(\omega)|^2$, estimated using RDD-FFT.*

Figure 5 and figure 4 shows the coherence functions for the FRFs shown in the above figures.

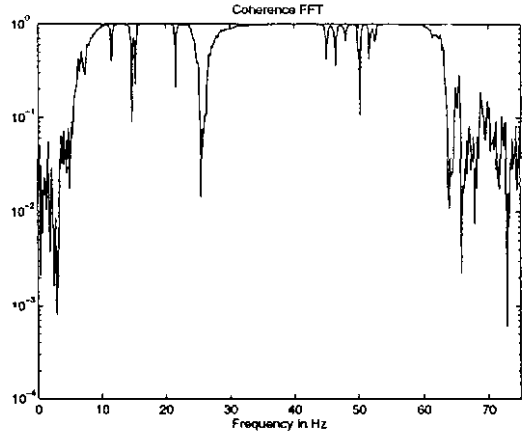

Figure 4: *Typical coherence function estimated using pure FFT.*

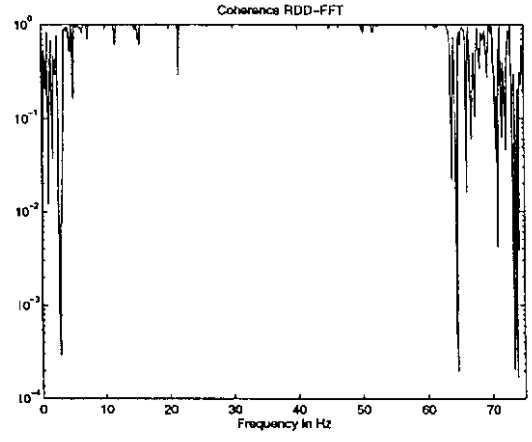

Figure 5: *Typical coherence function estimated using RDD-FFT.*

The estimated modal parameters are printed in table 1.

F [Hz]	1	2	3	4
RDD-FFT	11.67	15.50	21.53	45.09
FFT	11.70	15.51	21.54	45.11
F [Hz]	5	6	7	8
RDD-FFT	47.99	50.19	51.75	61.60
FFT	47.99	50.17	51.78	61.60
ζ [%]	1	2	3	4
RDD-FFT	0.005	0.001	0.001	0.001
FFT	0.010	0.009	0.002	0.001
ζ [%]	5	6	7	8
RDD-FFT	0.002	0.004	0.005	0.006
FFT	0.003	0.003	0.003	0.003

Table 1: *Estimated eigenfrequencies F in* Hz *and estimated damping ratios ζ in %*

There is a high correlation between the estimated eigenfrequencies, whereas the damping ratios generally are smaller for the RDD based modes. Some of the mode shapes are plotted in figure 6 - figure 11 which also includes the MAC between the modes.

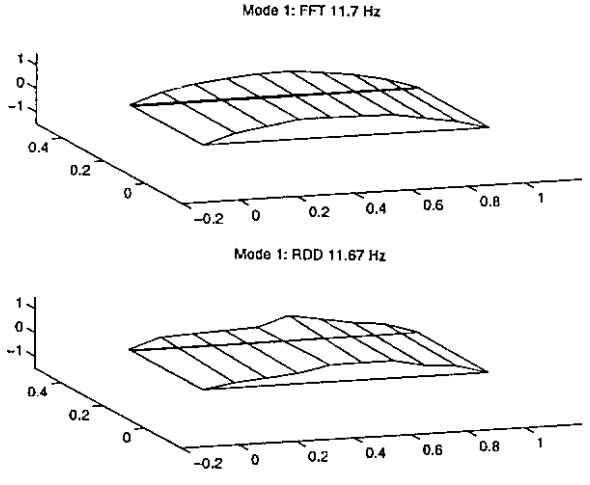

Figure 6: *First mode estimated using FFT and RDD-FFT. MAC=0.90.*

The difference in the two estimated modes of figure 6 is systematic and insensitive to the model order or the number of points used in the modal parameter extraction procedure

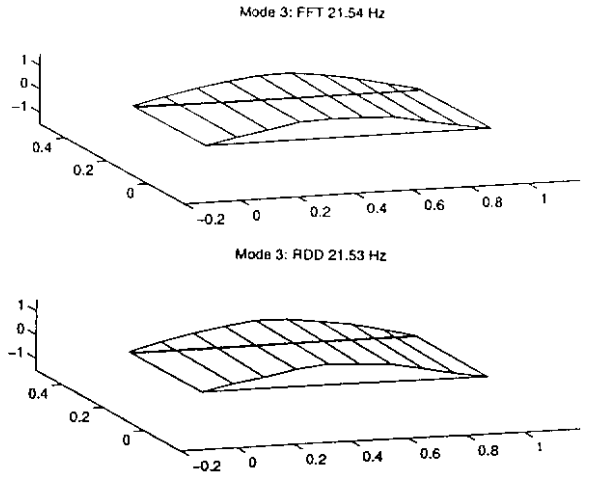

Figure 7: *Third mode estimated using FFT and RDD-FFT. MAC=0.9970.*

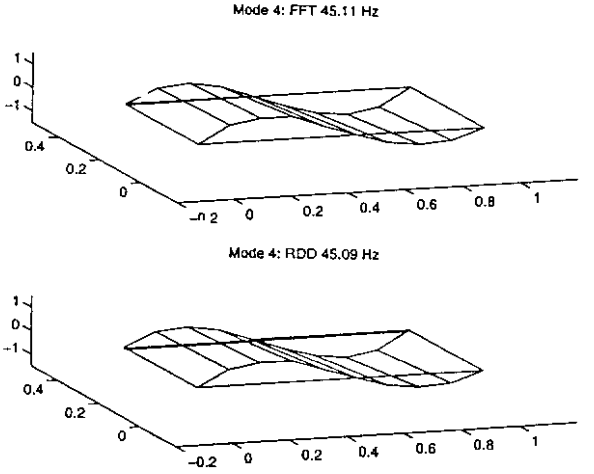

Figure 8: *Fourth mode estimated using FFT and RDD-FFT. MAC=0.9952.*

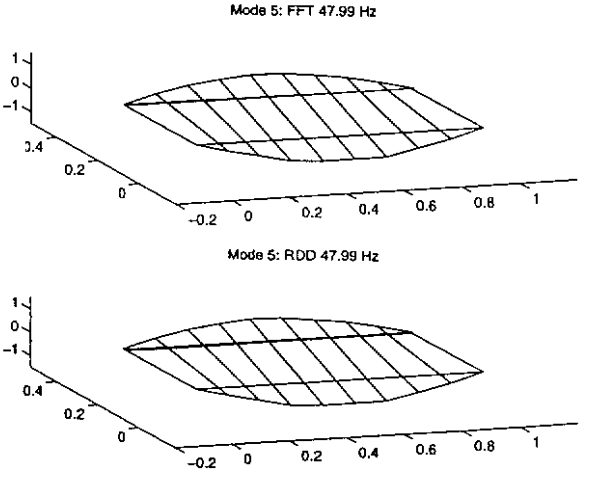

Figure 9: *Fifth mode estimated using FFT and RDD-FFT. MAC=0.9993*

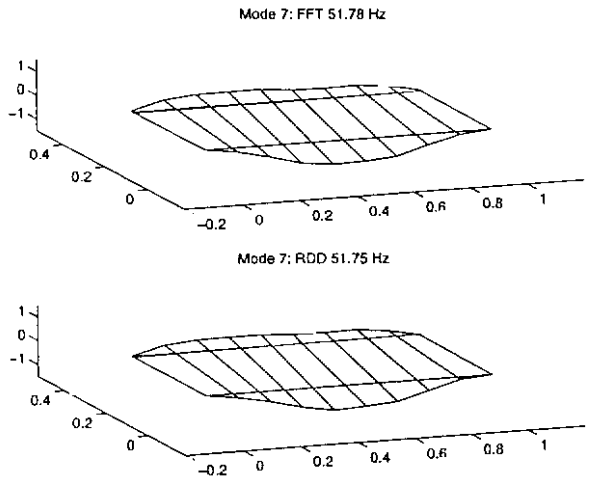

Figure 10: *Seventh mode estimated using FFT and RDD-FFT. MAC=0.9963.*

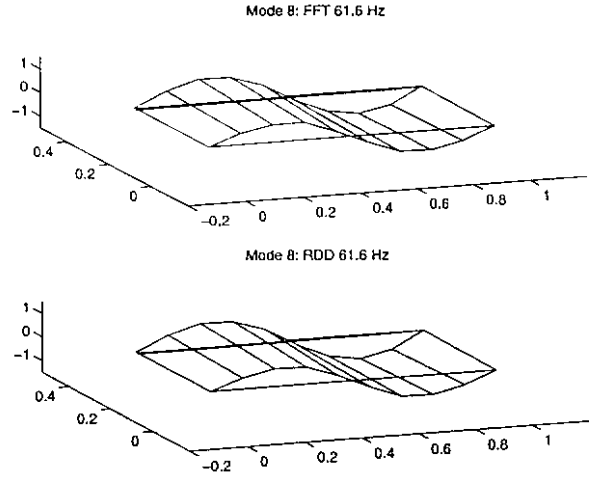

Figure 11: *Eighth mode estimated using FFT and RDD-FFT. MAC=0.9983.*

Table 2 illustrates the advantage of the RDD-FFT approach with respect to the estimation time. This approach is 3 times faster than the traditional FFT approach for estimating the FRFs of a single setup, consisting of 8 measurements of each 32000 points.

	RDD	FFT
Time [CPU]	82	240

Table 2: *Estimation times for RDD-FFT and pure FFT based FRFs.*

7 Conclusion

The principle of Random Decrement based FRF estimation is tested on data collected from a white noise loaded laboratory bridge model. The results are compared with the results from a traditional FFT based FRF estimation.

In general the application of both methods resulted in high-correlated modal parameters. The pure FFT approach is the most reliable approach but it is also considerably slower. Care must be taken in the choice of triggering condition and triggering levels with the RDD technique.

8 Acknowledgement

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