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An Engineering Interpretation of the Complex Eigensolution of Linear Dynamic Systems

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NOMENCLATURE

$\mathbf{A}$	state space system matrix
$\mathbf{B}$	state space load matrix
$\mathbf{M}$	mass matrix
$\mathbf{C}$	damping matrix
$\mathbf{K}$	stiffness matrix
$\mathbf{q}(t)$	vector of generalised time dependent displacements
$\dot{\mathbf{q}}(t)$	vector of generalised time dependent velocities
$\ddot{\mathbf{q}}(t)$	vector generalised time dependent accelerations
$\mathbf{Q}(t)$	time varying vector of nodal loads
$\mathbf{x}(t)$	state vector
$\dot{\mathbf{x}}(t)$	state vector time derivative
$\mathbf{x}$	right (column) state space eigenvector
$\mathbf{y}$	left (column) state space eigenvector
$\mathbf{X}$	matrix of right state space eigenvectors
$\mathbf{Y}$	matrix of left state space eigenvectors
t	time variable
t_0	initial time
$a_j(t_0)$	is the initial complex modal coordinate of mode *j* corresponding to the eigenvalue pair λ_j, λ_j^* and the initial state conditions $\mathbf{x}(t_0)$
λ	complex eigenvalue
θ_{jk}	is the phase of component *k* of right state space eigenvector *j*
ω_j	is the damped circular natural frequency of mode *j*
α_j	is the damping factor of mode *j*
^{T}	Super script ^{T} indicates transpose of a matrix
	$i = \sqrt{-1}$
x_{jk}	is the complex component *k* of right state space eigenvector *j*
Λ	diagonal matrix of complex eigenvalues
φ_j	is the initial modal phase of mode *j* i.e. $\varphi_j = \arg(a_j(t_0))$
ω_{0j}	is the undamped circular natural frequency of mode *j*
ζ_j	is the damping ratio of mode *j*
*	Superscript * indicates complex conjugate

ABSTRACT

In traditional finite element based modal analysis of linear non-conservative structures, the modal shapes are determined solely based on stiffness and mass. Damping effects are included by implicitly assuming that the damping matrix can be diagonalized by the undamped modes. The approach gives real valued mode shapes and modal coordinates. While this framework is suitable for analysis of most lightly damped structural systems, it is insufficient for interpretation of the free vibration and resonant response of structures with e.g. significant non-classical damping, gyroscopic or other effects resulting in a complex eigensolution. In this paper, the more general approach based on complex eigenvalues and eigenvectors is employed. We give an interpretation of the complex eigensolution that describes free and resonant vibrations of a generally damped linear structure. The interpretation show how the different parts of the complex eigensolutions; i.e. the complex left and right eigenvectors together with the complex eigenvalues, combines into vibration frequencies and modal damping ratios, mode shape magnitudes and phase angles, and modal coordinate magnitude and phase angles. The presented interpretation relates all elements of the complex valued solution to physical quantities that are well known in structural dynamics as well as other fields studying linear dynamic systems, and complements the already applied interpretations.

INTRODUCTION

The dynamics of linear structures are traditionally interpreted in terms of classical normal modes. Classical normal modes are defined as the modes belonging to linear undamped systems. T.K. Caughey [1] has also shown that a special class of damped systems have classical normal modes. A necessary and sufficient condition for a damped system to possess classical normal modes is that the damping matrix can be diagonalized by the transformation that uncouples the associated undamped system. We will refer to such systems as being classically damped. Real structures will generally not have a damping matrix that strictly satisfies the requirements for the system to possess classical normal modes. Nevertheless, for practical engineering purposes and lightly damped systems the normal mode approximation may be sufficiently good. However, the small damping assumption does not hold for e.g. deep-water risers. The dynamic behaviour of such structures exhibits progressive waves that cannot appear in a traditional normal mode approach.

In evaluation of the dynamic behaviour of structures, experiments have been widely applied to determine the dynamic properties. Since a modal analysis reveals the basic dynamic behaviour, it is a preferred coordinate basis for interpretation of measured dynamic response.

Today almost all measurements of structural response are processed by digital computers, yielding a time, amplitude and space discretized representation of the response. The discretization in time caused by the sampling process and the possibly imperfect time synchronisation between different measuring devices may introduce phase modulation into the vector of measured response time histories. Response measurements should therefore be processed and interpreted as coming from a system that permits spatial phase variations, even for cases that in reality are classically damped.

Thus, there is a need for an interpretation of the complex eigensolution of generally damped systems in terms of physical quantities such as mode shapes, vibration amplitudes and phase angles.

Several textbook authors have treated elements of this topic over the years; see e.g. Hurty and Rubinstein [2], Newland [3], Meirovitch [4] and Ewins [5]. The topic has also been treated or touched in several papers presented on IMAC conferences over the years, e.g. [6 - 18]

Hurty and Rubinstein show how to apply the complex eigensolution to obtain a real-valued forced response. However, they do not give an explicit interpretation of the complex eigensolution for the free and resonant vibration case.

Newland interprets the complex eigenvectors as counter rotating phasors, but does not show how they in fact combine into real valued response. The phase shift between elements of the mode shape belonging to different positions on the structure is briefly indicated in his presentation.

Meirovitch as well as Ewins presents the complex eigensolution, but does not give a complete physical interpretation in terms of mode shapes, modal amplitudes and corresponding phase angle of the complex quantities that constitutes the complex eigensolution.

In this paper, we will give a physical interpretation of the complex eigensolution that decouples a linear, spatially discretized, dynamic structural system, with general, not necessarily symmetric, mass, damping and stiffness properties. We interpret the modal shape as the envelopes and the spatial phase shifts determined by the magnitude and the phase angle of the corresponding normalized complex right eigenvector. We will show that for a damped free vibration, the initial condition given by a state vector containing generalized displacements and velocities, and the complex left eigenvectors of the system uniquely define the initial complex modal coordinates. The initial complex modal coordinate is interpreted as the amplitudes and the phase angles of the modal vibrations.

Realizing that each point in a response time series is the initial condition of an ensuing free vibration, one can also compute time series of modal amplitudes and phase angles from vector time series of response. This requires the left eigenvectors of the system to be known.

THE EQUATIONS OF MOTION

The damped free vibration of a linear time-invariant multi-degree-of-freedom structural system can be approximated by a spatially discrete second order differential equation as follows

$$\mathbf{M}\ddot{\mathbf{q}}(t)+\mathbf{C}\dot{\mathbf{q}}(t)+\mathbf{K}\mathbf{q}(t)=\mathbf{Q}(t) \tag{1}$$

where $\mathbf{Q}(t)$ is the time varying vector of nodal loads. The time varying generalised displacements vector is $\mathbf{q}(t)$, the generalised velocities vector is $\dot{\mathbf{q}}(t)$ and the generalised acceleration vector is $\ddot{\mathbf{q}}(t)$. The mass matrix $\mathbf{M}$ is assumed positive definite. The damping matrix $\mathbf{C}$ may contain both viscous damping terms and gyroscopic terms. Gyroscopic terms may occur for e.g. risers with internal flow, and likewise for towed cables, se e.g. Blevins [19], and of course for rotating shafts etc. Thus, the damping matrix may be non-symmetric. The stiffness matrix $\mathbf{K}$ may contain general stiffness properties. Normally the stiffness matrix will be symmetric. However, in certain flow-induced vibration problems, e.g. the classical flutter problem of airfoils, the equation of motion may be formulated to yield a non-symmetric stiffness matrix.

In the case of interpreting system matrices identified or estimated from measured response, i.e. system identification, the system matrices cannot be assumed symmetric even if the tested system should yield symmetric matrices in theory. One major reason for non-symmetry in the identified matrices is that measurements always are imperfect and noisy.

Thus, only under very special circumstances the eigenvalue problem of a system given by (1) will become symmetric and positive definite and thereby having real eigenvectors. In the general case complex eigenvectors occur.

The eigenvalue problem corresponding to (1) can be solved in two ways, either by direct solution of the corresponding quadratic eigenvalue problem or as will be done here, by recasting (1) into a first order system in state space form. The state space model is a robust and good engineering model with a good numerical foundation for treating linear vibrating systems and it is as easy to understand as the second order approach.

We begin by defining the state vector as a combination of the generalised configuration vector and the generalised velocity vector.

$$\mathbf{x}(t)=\begin{bmatrix}\mathbf{q}(t)\\ \dot{\mathbf{q}}(t)\end{bmatrix} \tag{2}$$

Then the second order differential equation, (1), can be recast into a first order system as shown below

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{Q}(t) \tag{3}$$

in which the system matrix $\mathbf{A}$ which is a $2n \times 2n$ real non-symmetric matrix, and the state-space load matrix $\mathbf{B}$ are defined by

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1} \end{bmatrix} \tag{4}$$

Other forms are also possible depending on the definition of the state vector and the properties of the matrices $\mathbf{M}$,$\mathbf{C}$ and $\mathbf{K}$. See e.g. Hurty and Rubinstein [2] or Laub and Arnold [20]. However, choice of formulation is only a matter of importance with respect to numerical implementation. They will all be related by simple coordinate transformations.

FREE VIBRATION

We will now investigate the damped free vibration case, i.e. the loading will be neglected: $\mathbf{Q}(t) = \mathbf{0}$. The free vibration solution is important because it can be applied directly in interpretation of the resonant response of the system. For the free vibration case, the solution of (3) has the exponential form

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{x} \tag{5}$$

In (5) λ is a scalar constant and $\mathbf{x}$ is a constant 2*n* vector. Both λ and $\mathbf{x}$ are in general complex valued. By inserting (5) into (3) and dividing through with $e^{\lambda t}$, we obtain the algebraic eigenvalue problem

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \tag{6}$$

The equation has 2*n* solutions in the form of eigenvalues λ_j and right eigenvectors $\mathbf{x}_j$, $\left(j = 1,2,\ldots\ldots,2n\right)$. The adjoint system admits the determination of the left eigenvectors $\mathbf{y}$

$$\mathbf{A}^T\mathbf{y} = \lambda\mathbf{y} \tag{7}$$

The justification of the name left eigenvector for $\mathbf{y}$ follows by transposing (7). It is easily seen that (6) and (7) have the same eigenvalues because $\det\mathbf{A}^T = \det\mathbf{A}$ so that $\det\left(\mathbf{A}^T - \lambda_i\mathbf{I}\right) = \det\left(\mathbf{A} - \lambda_i\mathbf{I}\right)$.

We now introduce the diagonal matrix of eigenvalues

$$\mathbf{\Lambda} = \mathrm{diag}\left(\lambda_j\right) \tag{8}$$

as well as the matrices of left and right column eigenvectors

$$\mathbf{Y} = \left[\mathbf{y}_1 \quad \mathbf{y}_2 \quad \ldots\ldots \quad \mathbf{y}_{2n}\right], \quad \mathbf{X} = \left[\mathbf{x}_1 \quad \mathbf{x}_2 \quad \ldots\ldots \quad \mathbf{x}_{2n}\right] \tag{9}$$

Assume for simplicity that all eigenvalues of $\mathbf{A}$ are distinct.

Left-multiply (6) by $\mathbf{y}_j^T$, right-multiply the transpose of (7) by $\mathbf{x}_j$ and subtract, to obtain

$$\left(\lambda_i - \lambda_j\right)\mathbf{y}_j^T \mathbf{x}_j = 0 \tag{10}$$

For $i \neq j$ and since the eigenvalues are assumed distinct, i.e. $\lambda_i \neq \lambda_j$, we must have

$$\mathbf{y}_j^T \mathbf{x}_i = 0 \tag{11}$$

i.e. the columns of $\mathbf{X}$ and $\mathbf{Y}$ are orthogonal.
It should be noticed that the assumption of distinct eigenvalues is a sufficient, but not necessary condition for orthogonality of the left and right eigenvector matrices. There exist cases where coinciding eigenvalues may possess two or more linearly independent eigenvectors; i.e. the geometric multiplicity of the eigenvalue is larger than one. One simple example of such a system is a beam with doubly symmetric cross section.

Left multiplying (6) by $\mathbf{y}_j^T$ and considering (11) we obtain

$$\mathbf{y}_j^T \mathbf{A} \mathbf{x}_i = 0\,, \quad \lambda_i \neq \lambda_j \tag{12}$$

After normalising the left and right eigenvectors by requiring

$$\mathbf{y}_i^T \mathbf{x}_i = 1 \tag{13}$$

we obtain

$$\mathbf{y}_i^T \mathbf{A} \mathbf{x}_i = \lambda_i \tag{14}$$

Thus, the two matrices $\mathbf{X}$ and $\mathbf{Y}$ satisfy the biorthonormality relations both with respect to each other and with respect to the matrix $\mathbf{A}$. This is expressed in compact matrix form as follows

$$\begin{aligned} \mathbf{Y}^T \mathbf{X} &= \mathbf{I} \\ \mathbf{Y}^T \mathbf{A} \mathbf{X} &= \mathbf{\Lambda} \end{aligned} \tag{15}$$

The first of these equations implies

$$\mathbf{Y}^T = \mathbf{X}^{-1} \tag{16}$$

Then we obtain the following expression for the eigenvalue matrix

$$\mathbf{X}^{-1} \mathbf{A} \mathbf{X} = \mathbf{\Lambda} \tag{17}$$

We see that the complex eigenvector matrices decouple the equation system. Equation (17) represents a similarity transformation and the matrices $\mathbf{A}$ and $\mathbf{\Lambda}$ are said to be similar. It is also well known that for a system with orthogonal eigenvectors, the eigenvalues do not change under similarity transformations. Thus, any realisation of $\mathbf{A}$ for the same underlying system will have the same eigenvalues.

The system described by the matrix $\mathbf{A}$ is said to be a stable system if the real parts of all the eigenvalues in $\mathbf{\Lambda}$ are strictly less than zero. This is assumed in the following.

INTERPRETATION OF THE COMPLEX EIGENSOLUTION

Consider the free vibration problem, i.e. (3) with $\mathbf{Q}(t)=\mathbf{0}$ and its solution given by (5). This solution represents the response to the state initial conditions, $\mathbf{x}(t_0)$, which specifies the initial displacement and velocity of the system. Physical reasons imply that the response must be a real quantity. The solution of the free vibration problem can be expressed as a linear combination of 2*n* independent solutions

$$\mathbf{x}(t)=\mathbf{X}e^{\mathbf{\Lambda}(t-t_0)}\mathbf{a}(t_0) \tag{18}$$

where

$$\mathbf{a}(t_0)=\begin{bmatrix}a_1(t_0) & a_2(t_0) & \ldots\ldots & a_{2n}(t_0)\end{bmatrix}^T$$

$$e^{\mathbf{\Lambda}(t-t_0)}=\mathrm{diag}\begin{bmatrix}e^{\lambda_1(t-t_0)} & e^{\lambda_2(t-t_0)} & \ldots\ldots & e^{\lambda_{2n}(t-t_0)}\end{bmatrix}$$

$\mathbf{a}(t_0)$ is a vector of complex coefficients combining the eigenvectors, i.e. $\mathbf{a}(t_0)$ contains the complex modal coordinates. The complex modal coordinates $\mathbf{a}(t_0)$ are related to the specific initial condition $\mathbf{x}(t_0)$ as will be shown in the following.

Setting $t=t_0$, pre-multiplying both sides of (18) with $\mathbf{Y}^T$ and invoking the orthonormality conditions (15), one obtains

$$\mathbf{a}(t_0)=\mathbf{Y}^T\mathbf{x}(t_0) \tag{19}$$

The elements of the initial modal coordinate vector $\mathbf{a}(t_0)$ can therefore be written as

$$a_j(t_0)=\mathbf{y}_j^T\mathbf{x}(t_0) \tag{20}$$

The initial complex modal coordinates, $a_j(t_0)$, are thus computed by utilising the orthonormality properties of the left and right eigenvectors of the system matrix. The magnitude and the phase angle of the complex initial modal coordinate interpret as the initial modal amplitude and the initial modal phase angle.

We recall that each point in a response time series may be considered as the initial conditions for an ensuing free vibration. From (20) it follows directly that the time series of complex modal coordinates of state space mode *j* corresponding to the eigenvalue pair λ_j, λ_j^* can be computed for any given response time series provided the corresponding left eigenvector is known. Especially this is a useful result for interpretation of measured response from structures excited by random load processes.

By applying (19), the free vibration solution (18) can now be rewritten as

$$\mathbf{x}(t)=\mathbf{X}e^{\mathbf{\Lambda}(t-t_0)}\mathbf{Y}^T\mathbf{x}(t_0), \quad t\geq t_0 \tag{21}$$

The right hand side of (21) must be real, since $\mathbf{x}(t)$ is real. Thus for every complex eigenvalue λ_j of $\mathbf{\Lambda}$ there must be a complex conjugate eigenvalue λ_j^* to ensure that the imaginary parts of the response cancel at all times. Provided λ_j is not real, then the eigenvectors will also be non-real, since an assumption of real eigenvectors would yield real left hand sides of (6) and (7), while the right hand sides would be non-real, i.e. a contradictory result. Furthermore by taking the complex conjugate of (6) and (7), it is seen that the eigenvectors to λ_j^* must be $\mathbf{x}_j^*$ and $\mathbf{y}_j^*$. Each complex eigenvector and its corresponding complex conjugate represent the response of a sub-critically damped linear oscillator. Purely imaginary eigenvalues can only occur if the system has no damping. Some or all eigenvalues in $\mathbf{\Lambda}$ may be real, representing the response of an over-damped system.

Assume for simplicity of notation that $\mathbf{\Lambda}$ contains only complex eigenvalues. We will then have *n* pairs of eigenvalues $(\lambda_j,\lambda_j^*),(j=1,2,\ldots\ldots,n)$. The response can then be expressed as the following sum over *n* components

$$\begin{aligned}\mathbf{x}(t)&=\sum_{j=1}^{n}\left(\mathbf{x}_j e^{\lambda_j(t-t_0)}\mathbf{y}_j^T\mathbf{x}(t_0)+\mathbf{x}_j^* e^{\lambda_j^*(t-t_0)}\mathbf{y}_j^{*T}\mathbf{x}(t_0)\right)\\&=\sum_{j=1}^{n}\left(\mathbf{x}_j e^{\lambda_j(t-t_0)}a_j(t_0)+\mathbf{x}_j^* e^{\lambda_j^*(t-t_0)}a_j^*(t_0)\right)\end{aligned} \tag{22}$$

Consider the polar form of the complex numbers in the above equation

$$\begin{aligned}&e^{\lambda_j(t-t_0)}=e^{-\alpha_j(t-t_0)}\cdot e^{i\omega_j(t-t_0)}\\&x_{jk}=|x_{jk}|e^{i\theta_{jk}}, \qquad \theta_{jk}=\arg(x_{jk})\\&a_j(t_0)=|a_j(t_0)|e^{i\phi_j(t_0)}, \qquad \phi_j(t_0)=\arg(a_j(t_0))\\[1em]&e^{\lambda_j^*(t-t_0)}=e^{-\alpha_j(t-t_0)}\cdot e^{-i\omega_j(t-t_0)}\\&x_{jk}^*=|x_{jk}|e^{-i\theta_{jk}}, \qquad -\theta_{jk}=\arg(x_{jk}^*)\\&a_j^*(t_0)=|a_j(t_0)|e^{-i\phi_j(t_0)}, \qquad -\phi(t_0)=\arg(a_j^*(t_0))\end{aligned} \tag{23}$$

Substituting (23) for the complex numbers in (22) the following expression for element *k* of the free vibration response vector is obtained

$$\begin{aligned}x_k(t)&=\sum_{j=1}^{n}\left(|x_{jk}|e^{i\theta_{jk}}e^{-\alpha_j(t-t_0)}e^{i\omega_j(t-t_0)}|a_j(t_0)|e^{i\phi_j(t_0)}+|x_{jk}|e^{-i\theta_{jk}}e^{-\alpha_j(t-t_0)}e^{-i\omega_j(t-t_0)}|a_j(t_0)|e^{-i\phi_j(t_0)}\right)\\&=\sum_{j=1}^{n}\left(2\,|a_j(t_0)|\,|x_{jk}|e^{-\alpha_j(t-t_0)}\cos\left(\omega_j(t-t_0)+\theta_{jk}+\phi_j(t_0)\right)\right), \quad (k=1,2,\ldots,2n), t\geq t_0\end{aligned} \tag{24}$$

The quantities that appear in (24) interpret as follows:

$2\lvert a_j(t_0)\rvert$	is the initial modal amplitude of mode *j* corresponding to the eigenvalue pair λ_j, λ_j^* and the initial state conditions $\mathbf{x}(t_0)$
$\lvert x_{jk}\rvert$	is the magnitude of component *k* of right state space eigenvector *j*
$\alpha_j = \zeta_j \omega_{0j}$	is the damping factor of mode *j*
$\omega_j = \omega_{0j}\sqrt{1-\zeta^2}$	is the damped circular natural frequency of mode *j*
$\theta_{jk} = \arg(x_{jk})$	is the phase of component *k* of right state space eigenvector *j*
$\varphi_j = \arg(a_j(t_0))$	is the initial modal phase of mode *j* corresponding to the eigenvalue pair λ_j, λ_j^* and the initial state conditions $\mathbf{x}(t_0)$
ω_{0j}	is the undamped circular natural frequency of mode *j*
ζ_j	is the damping ratio of mode *j*

Thus, a generally damped linear structural system decouples into *n* state space modes, each with 2*n* components corresponding to generalised displacements and velocities. The state space modes are defined by means of the complex eigenvectors of the system containing magnitudes and phase angles. The appearance of spatially varying phase angles admits travelling wave behaviour of the mode shape as the oscillation proceeds through a cycle. This is a major and important difference from the synchronous standing oscillation found for classically damped systems.

Furthermore, the complex modal coordinates which determine the modal amplitude and the modal phase can be defined by means of the left eigenvectors and the state vector at time t_0, i.e. the initial condition. This is obtained because of the biorthonormality properties of the complex eigenvectors and the system matrices.

CONCLUSIONS

It has been shown how the complex eigensolution of general linear non-conservative dynamic systems can be interpreted in terms of mode shapes given by envelopes and phase shifts that are function of the position coordinate.

Furthermore, it has been shown how the left eigenvectors of the system and the initial state combine to yield the initial state modal coordinate. The initial state modal amplitude is the magnitude of the initial state modal coordinate, while the initial state modal phase is the phase angle of the initial state modal coordinate.

By treating each point in a response vector time series as the initial condition for an ensuing free vibration, time series of the modal coordinates are easily computed by the established definition of modal coordinates. This requires that the left eigenvectors of the system be known. However, these are easily computed using the definition of the adjoint system.

The established definitions are very useful tools for interpretation and understanding results obtained from system identification applied to measured and simulated resonant response of dynamic systems. This is especially the case for systems with non-classical damping and/or non-symmetries in the system matrices as will occur in e.g. several flow-induced vibration problems. An example of such an application is found in [21].

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