Some Methods to Determine Scaled Mode Shapes in Natural Input Modal Analysis

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NOMENCLATURE

Scaling factor	α
Scaled mode shape	$\{\phi\}$
Un-scaled mode shape	$\{\psi\}$
Natural frequency of the unmodified structure	ω_0
Scaled mode shape of the unmodified structure	$\{\phi\}_0$
Un-scaled mode shape of the unmodified structure	$\{\psi\}_0$
Natural frequency of the modified structure	ω_1
Scaled mode shape of the modified structure	$\{\phi\}_1$
Un-scaled mode shape of the modified structure	$\{\psi\}_1$
Stiffness matrix	$[k]$
Mass matrix	$[m]$
Damping matrix	$[c]$
Mass change matrix	$[\Delta m]$

ABSTRACT

When the modal model is going to be used for structural modification or for structural response simulation, the scaled mode shapes must be known. If natural input modal analysis is performed, only un-scaled mode shapes can be obtained and an extra method is necessary to obtain the scaling factor. In this paper, two new methods based on mass change are proposed. The first method involves small mass changes in two repeated tests allowing to achieve good accuracy. The second method involves only one mass change and enables the scaling factors of both the modified and unmodified mode shapes to be obtained. Finally, the effect of the normalization used in the mode shapes and the accuracy of each method are analyzed by simulation.

1. INTRODUCTION

In the last years, natural input modal analysis has become a powerful technology which can be applied to a wider range of the structures [**1**]. If the identified modal model is going to be used for structural response simulation, structural modification or health monitoring, then the scaled mode shapes must be known. When the calculation of frequency response function (FRF) or the impulse response function (IRF) are obtained from modal parameters, the following information is needed for each mode:

- The natural frequency ω,
- The damping factor ζ,
- The mass normalized (scaled) mode shape $\{\phi\}$.

However, when natural input modal analysis is performed, the forces are unknown and for this reason only the following information is obtained for each mode:

- The natural frequency ω,
- The damping factor ζ,
- The un-scaled mode shape $\{\psi\}$.

Therefore, an extra method to calculate the scaling factors α is needed. In general, these methods are based on a more extensive experimental testing procedure consisting of two stages:

1. Performing Natural input modal analysis to obtain the modal parameters ω, ζ and $\{\psi\}$.
2. Extra experimental testing to calculate the scaling factors α.

Even though the scaling factors can be obtained from the mode shapes and the mass matrix (or the stiffness matrix), this method is, in general, unpractical due to the fact that these matrices are difficult to estimate. Further, only for small structures, static test can be easily applied to estimate the stiffness matrix.

Recently, some methods have been proposed to estimate the scaling factors which involve repeated testing in which mass changes are introduced in the points where the mode shapes are known [1] [2] [3]. In this paper, two new methods involving mass change are proposed to estimate the scaling factors of the mode shapes. The first method involves small mass changes in two repeated tests with which good accuracy can be achieved. The second method involves mass change once only. The method provides exact solutions when the number of degree of freedoms is equal to the number of modes. If the number of modes is less than the number of degree of freedoms, the accuracy depends on the mode, the mass change and the truncation level of the modal model.

2. THE SCALING FACTOR

The scaled and the un-scaled mode shape are related by the equation:

$$\{\phi\} = \frac{\{\psi\}}{\sqrt{\{\psi\}^T \cdot [m] \cdot \{\psi\}}} \qquad (1)$$

so that the scaling factor is expressed as:

$$\alpha = \frac{1}{\sqrt{\{\psi\}^T \cdot [m] \cdot \{\psi\}}} \qquad (2)$$

and

$$\{\phi\} = \alpha \cdot \{\psi\} \qquad (3)$$

Alternatively, the stiffness matrix and the natural frequencies can be used, so that the scaling factor can be calculated from:

$$\alpha = \frac{\omega}{\sqrt{\{\psi\}^T \cdot [k] \cdot \{\psi\}}} \tag{4}$$

In the following paragraphs it will be assumed that the damping is proportional, so that the modes will be real.

3. THE MASS CHANGE METHOD.

The scaling factor estimation by the mass change method involves repeated testing where mass changes are introduced in the points of the structure where the mode shape is known [1] [3]. In the following paragraphs three methods to estimate the scaling factors based on mass change are proposed.

4.1 A simple method.

With this method [1] [3], the equation to estimate the scaling factor is derived from the basic equation of motion of a structure subjected to a force $\{p(t)\}$,i.e.:

$$[m] \cdot \{\ddot{u}\} + [c] \cdot \{\dot{u}\} + [k] \cdot \{u\} = \{p(t)\}\,. \tag{5}$$

The classical eigenvalue equation in case of no damping or proportional damping is:

$$[m] \cdot \{\phi_0\} \cdot \omega_0^2 = [k] \cdot \{\phi_0\}, \tag{6}$$

where $\{\phi_0\}$ is the mode shape, ω_0 the natural frequency, $[m]$ the mass matrix and $[k]$ the stiffness matrix. If we make a mass change so that the new mass matrix is $[m] + [\Delta m]$, the eigenvalue equation becomes:

$$\left([m] \;\; + [Am] \;\; \right) \cdot \{\phi_1\} \cdot \omega_1^2 = [k] \;\; \cdot \{\phi_1\}, \tag{7}$$

where $\{\phi_1\}$and ω_1 are the new modal parameters of the modified problem.
Substracting equations (6) and (7) we obtain:

$$[m] \cdot \left(\{\phi_0\} \cdot \omega_0^2 - \{\phi_1\} \cdot \omega_1^2 \right) - [\Delta m] \cdot \{\phi_1\} \cdot \omega_1^2 = [k] \cdot \left(\{\phi_0\} - \{\phi_1\} \right). \tag{8}$$

If we now assume that the mass change is so small that the mode shapes does not change significantly, i.e.:

$$\{\phi_1\} \cong \{\phi_0\} \cong \{\phi \;\; \}, \tag{9}$$

we get the equation:

$$[m] \cdot \{\phi\} \cdot \left(\omega_0^2 - \omega_1^2\right) = [\Delta m] \cdot \{\phi\} \cdot \omega_1^2 \tag{10}$$

Premultiplying equation (10) by $\{\phi\}^T$ results in:

$$\{\phi\}^T \cdot [m] \cdot \{\phi\} \cdot \left(\omega_0^2 - \omega_1^2\right) = \{\phi\}^T \cdot [\Delta m] \cdot \{\phi\} \cdot \omega_1^2 \tag{11}$$

Now taking in account the orthogonality of the modes, i.e:

$$\{\phi\}^T \cdot [m] \cdot \{\phi\} = 1\,, \tag{12}$$

the equation (11) becomes:

$$\left(\omega_0^2-\omega_1^2\right)=\{\phi\}^T\cdot[\Delta m]\cdot\{\phi\}\cdot\omega_1^2 . \tag{13}$$

Finally, combining equation (3) and equation (13) we obtain:

$$\left(\omega_0^2-\omega_1^2\right)=\alpha^2\cdot\{\psi\}^T\cdot[\Delta m]\cdot\{\psi\}\cdot\omega_1^2 . \tag{14}$$

And the unknown scaling factor can be derived from [1]:

$$\alpha=\sqrt{\frac{\left(\omega_0^2-\omega_1^2\right)}{\omega_1^2\cdot\{\psi\}^T\cdot[\Delta m]\cdot\{\psi\}}} \tag{15}$$

When using equation (15) to determine a scaling factor, only the mode shape and the natural frequency of this particular mode has to be known. Equation (15) gives exact results when the matrix $[\Delta m]$ is proportional to the mass matrix $[m]$ because in this case the modes remain unmodified [1]. When the mass change is oddly distributed, the error is dependent on the mode.

In equation (15), both the modified and unmodified mode shapes can be used. However, the better results are obtained using the unmodified mode shapes, i.e.:

$$\alpha=\sqrt{\frac{\left(\omega_0^2-\omega_1^2\right)}{\omega_1^2\cdot\{\psi_0\}^T\cdot[\Delta m]\cdot\{\psi_0\}}} , \tag{16}$$

or both the unmodified and the modified mode shapes, i.e.:

$$\alpha=\sqrt{\frac{\left(\omega_0^2-\omega_1^2\right)}{\omega_1^2\cdot\{\psi_0\}^T\cdot[\Delta m]\cdot\{\psi_1\}}} . \tag{17}$$

The influence of the type of normalization used in the mode shapes will be studied in this paper. The results obtained with equation (16) are independent of the type of normalization used whereas normalization to the length should be used with equation (17).

4.2 The extrapolation method.

The exact scaling factor of the unmodified mode shape $\{\psi_0\}$ can be calculated by means of:

$$\alpha_0=\frac{1}{\sqrt{\{\psi_0\}^T\cdot[m]\cdot\{\psi_0\}}} . \tag{18}$$

However, equation (18) can not be applied because the mass matrix can not be, in general, obtained by modal analysis. If a mass change is performed, the new exact scaling factor of the modified mode shape will be:

$$\alpha_1=\frac{1}{\sqrt{\{\psi_1\}^T\cdot(\,[m]+[\Delta m]\,)\cdot\{\psi_1\}}} . \tag{19}$$

If we assume that the mode shape does not change significantly, i.e, $\{\psi_1\}\cong\{\psi_0\}$, the equation (19) becomes:

$$\alpha_1=\frac{1}{\sqrt{\{\psi_0\}^T\cdot[m]\cdot\{\psi_0\}+\{\psi_0\}^T\cdot[\Delta m]\cdot\{\psi_0\}}} \tag{20}$$

or:

$$\alpha_1 = \frac{1}{\sqrt{\frac{1}{\alpha_0^2} + \{\psi_0\}^T \cdot [\Delta m] \cdot \{\psi_0\}}} \qquad (21)$$

From equation (21) can be inferred that the scaling factor diminishes when a mass change is applied to the structure.
On the other hand, the limit of the equation (21) when $[\Delta m] \rightarrow [0]$ is given by:

$$\lim_{[\Delta m]\rightarrow[0]} \left(\frac{1}{\sqrt{\frac{1}{\alpha_0^2} + \{\psi_0\}^T \cdot [\Delta m] \cdot \{\psi_0\}}} \right) = \alpha_0^2 . \qquad (22)$$

Equation (22) suggests that making several mass changes $\beta_1 \cdot [\Delta m]$, $\beta_2 \cdot [\Delta m]$, …, $\beta_N \cdot [\Delta m]$, then calculating the corresponding N scaling factors α_1, α_2, …, α_N with equation (16) and finally plotting the scaling factors versus the mass change factors β (Figure 1), a very good approximation of the scaling factor α_0 can be achieved by extrapolating toward $\beta = 0$ the curve that fit the results.

Due to the fact that we have to perform mass changes several times, the method may result expensive for big structures. However, if the mass change is small, good results can be attained changing mass twice and extrapolating a straight line (Figure 2)

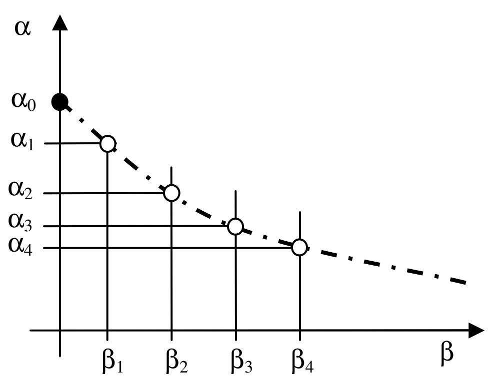

Fig. 1. The extrapolation method

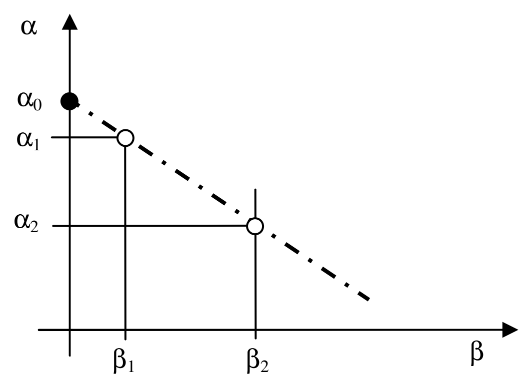

Fig 2. The extrapolation method with two points.

4.3 The coupled method.

Contrary to the two former mass change methods, in this method all modes are coupled, i.e, to determine a scaling factor, the natural frequencies and the mode shapes of all modes have to be used. The basis of the method is to apply modal decomposition twice, firstly to the basic equations of motion and secondly to the equations derived from the first modal decomposition.

In case of no damping or proportional damping, the equation of motion of a structure subjected to a force $\{p(t)\}$

$$[m] \cdot \{\ddot{u}\} + [c] \cdot \{\dot{u}\} + [k] \cdot \{u\} = \{p(t)\} , \qquad (23)$$

provides the eigenvalue equation:

$$[m]\cdot\{\phi_0\}\cdot\omega_0^2=[k]\cdot\{\phi_0\}. \quad (24)$$

The eigenvectors satisfy the orthonormal conditions :

$$[\Phi_0]^T\cdot[m]\cdot[\Phi_0]=[I]$$

$$[\Phi_0]^T\cdot[k]\cdot[\Phi_0]=\left[\omega_0^2\right] \quad (25)$$

$$[\Phi_0]^T\cdot[c]\cdot[\Phi_0]=\left[2\zeta_0\omega_0\right]$$

If a mass change resulting in a new mass matrix $[m]+[\Delta m]$ to undertaken, the new equation of motion becomes:

$$\left([m]+[\Delta m]\right)\cdot\{\ddot{u}\}+[c]\cdot\{\dot{u}\}+[k]\cdot\{u\}=\{p(t)\} \quad (26)$$

Using the transformation [4] (modal decomposition)

$$\{u\}=[\Phi_0]\cdot\{q\} \quad (27)$$

in the equation of motion (26), after premultiplying by $[\Phi_0]^T$ we obtain [4]:

$$[\Phi_0]^T\cdot\left([m]+[\Delta m]\right)\cdot[\Phi_0]\cdot\{\ddot{q}\}+[\Phi_0]^T\cdot[c]\cdot[\Phi_0]\cdot\{\dot{q}\}+[\Phi_0]^T\cdot[k]\cdot[\Phi_0]\cdot\{u\}=[\Phi_0]^T\cdot\{p(t)\}. \quad (28)$$

Taking in account the orthogonality conditions (25), the previous equation becomes:

$$\left([I]+[\Phi_0]^T\cdot[\Delta m]\cdot[\Phi_0]\right)\cdot\{\ddot{q}\}+[2\zeta_0\omega_0]\cdot\{\dot{q}\}+\left[\omega_0^2\right]\cdot\{q\}=[\Phi_0]^T\cdot\{p(t)\} \quad (29)$$

which provides the eigenvalue problem:

$$\left([\ I\ ]+[\Phi_0]^T\cdot[\Delta m]\cdot[\Phi_0]\right)\cdot\{\phi_b\}\cdot\omega_b^2=\left[\omega_0^2\right]\cdot\{\phi_b\}. \quad (30)$$

The natural frequencies ω_b^2 and the modes $\{\phi_b\}$ uncouple the equations of motion (29) by means of the transformation:

$$\{q\}=[\Phi_b]\cdot\{q_b\}. \quad (31)$$

Combining equations (27) and (31), results in:

$$\{u\}=[\Phi_0]\cdot[\Phi_b]\cdot\{q_b\} \quad (32)$$

which relates the physical coordinates with the modal coordinates of the modified structure. The modes of the modified structure $[\Phi_1]$ are given by [4]:

$$[\Phi_1]=[\Phi_0]\cdot[\Phi_b] \quad (33)$$

and the natural frequencies $\left[\omega_1^2\right]$ by:

$$\left[\omega_1^2\right] = \left[\omega_b^2\right] \ . \tag{34}$$

On the other hand, the scaled mode shapes of the unmodified structure can be expressed as:

$$[\Phi_0] = [\alpha_0]\cdot[\Psi_0] \ , \tag{35}$$

where the matrix

$$[\alpha_0] = \begin{bmatrix} \alpha_1 & \alpha_1 & \dots & \alpha_1 \\ \alpha_2 & \alpha_2 & \dots & \alpha_2 \\ \dots & \dots & \dots & \\ \alpha_N & \alpha_N & \dots & \alpha_N \end{bmatrix}_0 \tag{36}$$

contains the scaling factors corresponding to each mode.

The scaled mode shapes of the modified structure can also be expressed as:

$$[\Phi_1] = [\alpha_1]\cdot[\Psi_1] \ , \tag{37}$$

where the matrix

$$[\alpha_1] = \begin{bmatrix} \alpha_1 & \alpha_1 & \dots & \alpha_1 \\ \alpha_2 & \alpha_2 & \dots & \alpha_2 \\ \dots & \dots & \dots & \\ \alpha_N & \alpha_N & \dots & \alpha_N \end{bmatrix}_1 \tag{38}$$

contains the scaling factors corresponding to each mode of the modified structure.

From experimental modal analysis, the following information can be known:

$$\left[\omega_0^2\right]_{exp} \quad [\Psi_0]_{exp} \quad \left[\omega_1^2\right]_{exp} \quad [\Psi_1]_{exp} \tag{39}$$

together with the mass change matrix $[\Delta m]$. If we assume that

$$\begin{aligned} [\Psi_0] &= [\Psi_0]_{exp} \quad \left[\omega_0^2\right] = \left[\omega_0^2\right]_{exp} \\ [\Psi_1] &= [\Psi_1]_{exp} \quad \left[\omega_1^2\right] = \left[\omega_1^2\right]_{exp} \end{aligned} \tag{40}$$

the modified modes can be expressed as:

$$[\Phi_1] = [\alpha_0]\cdot[\Psi_0]_{exp}\cdot[\Phi_b] \tag{41}$$

The former equations suggest estimating the scaling factors minimizing the error between the estimated and experimental mode shapes and natural frequencies, i.e:

$$\begin{aligned} \varepsilon_1 &= [\Phi_1] - [\alpha_1]\cdot[\Psi_1]_{exp} = [\alpha_0]\cdot[\Psi_0]_{exp}\cdot[\Phi_b] - [\alpha_1]\cdot[\Psi_1]_{exp} \\ \varepsilon_2 &= \left[\omega_1^2\right]_{exp} - \left[\omega_b^2\right] \end{aligned} \tag{42}$$

with respect to $[\alpha_0]$and $[\alpha_1]$. The modes $[\Phi_b]$ and natural frequencies $\left[\omega_b^2\right]$ will be obtained solving the eigenvalue problem (30) .

Due to the fact that the system of equations (42) is non-linear, initial values of scaling factors are needed. Initial values of $[\alpha_0]$ and $[\alpha_1]$ can be obtained using equations (15) and (21), respectively. The algorithm of the process is shown in Figure 3.

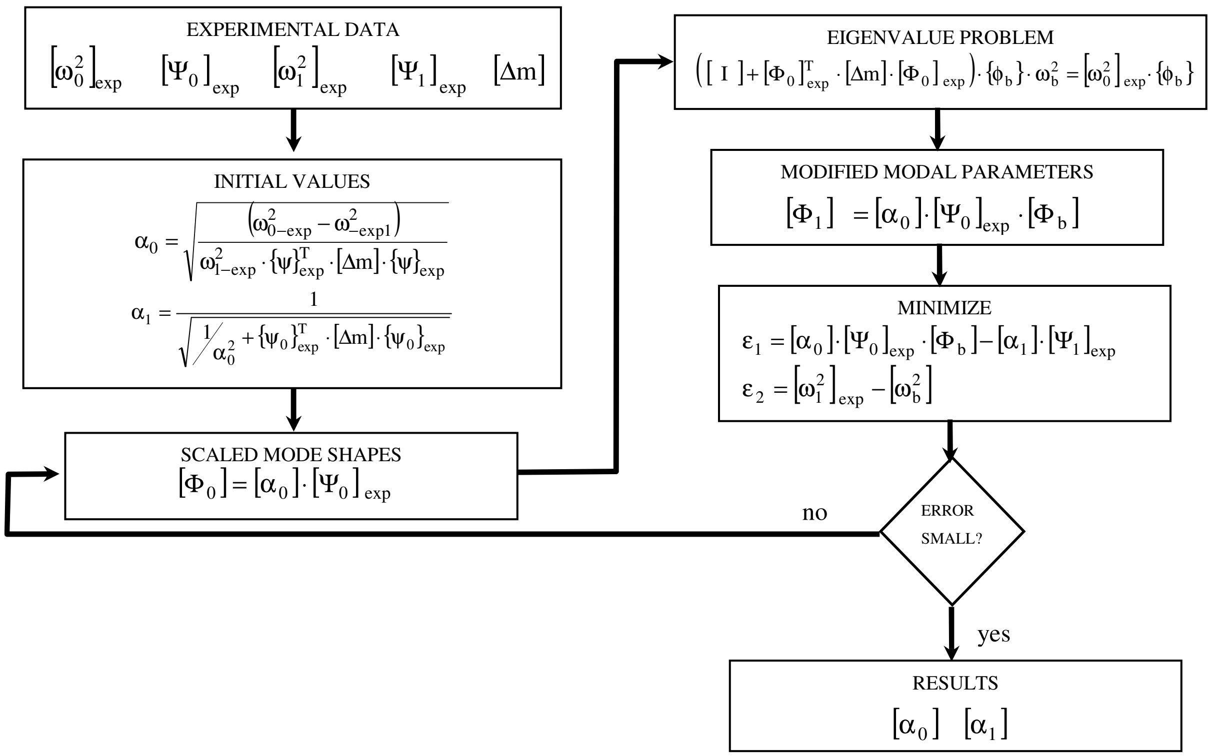

Fig 3. Algorithm of the coupled method.

One advantage of this method is that the error between the estimated and the experimental mode shapes can be known from equation (42). If the error in the estimated mode shape is high, the error in the estimated scaling factor will also be high.

The scaling factors determined with this method will be the exact ones when the number of observation points is equal to the number of modes. The method can also be applied in the same manner when the number of modes is less than the number of the observation points, but in this case the solution is only an approximation and the error depends on the mass change magnitude, the mass change position, the truncation level and the mode.

4. ILUSTRATION OF THE ACCURACY

The accuracy of the proposed methods are studied in the following sections. The accuracy obtained with each method depends on the type of normalization used in the mode shapes and this effect will be analyzed in the first section. In the second section, normalization to the length will be used and the accuracy of the proposed methods will be studied.

5.1 Influence of normalization

In this section the influence of the type of normalization used in the mode shapes will be studied. Two types of normalization will be analysed: normalization of one coordinate (the largest component equal 1) and normalization to the length (the length of the mode equal 1). The normalization to the length is performed by means of the expression:

$$\frac{\{\psi\}}{\sqrt{\{\psi\}^T \cdot \{\psi\}}} \tag{43}$$

The influence of normalization has been studied by performing one thousand simulations on a simple dynamic system:

$$[k] = k \begin{bmatrix} 2 & -1 & . & 0 & 0 \\ -1 & 2 & . & 0 & 0 \\ . & . & . & . & . \\ 0 & 0 & . & 2 & -1 \\ 0 & 0 & . & -1 & 2 \end{bmatrix} \quad [m] = m \begin{bmatrix} 1 & & . & 0 & 0 \\ 0 & 1 & . & 0 & 0 \\ . & . & . & . & . \\ 0 & 0 & . & 1 & 0 \\ 0 & 0 & . & & 1 \end{bmatrix} \tag{44}$$

The system has 20 degrees of freedom and only the first 5 modes were used in the investigation. The simulations were performed placing masses in 10 random points. The total mass change was 10% of the total mass.

The scaling factor was calculated using equations (16) and (17) and the extrapolation method. In the case of the extrapolation method, in which application two different mass changes have to be carried out, the magnitude of the first and of the second mass change were respectively $[\Delta m]$ and $2[\Delta m]$.

The errors on the scaling factor are shown in Figure 4. As can be seen, the results obtained with equation (16) and with the extrapolated method are independent of the type of normalization used. If normalization to the unity is performed, the best results are obtained with the extrapolation method. When normalization to the length is used, the best results are obtained with equation (17).

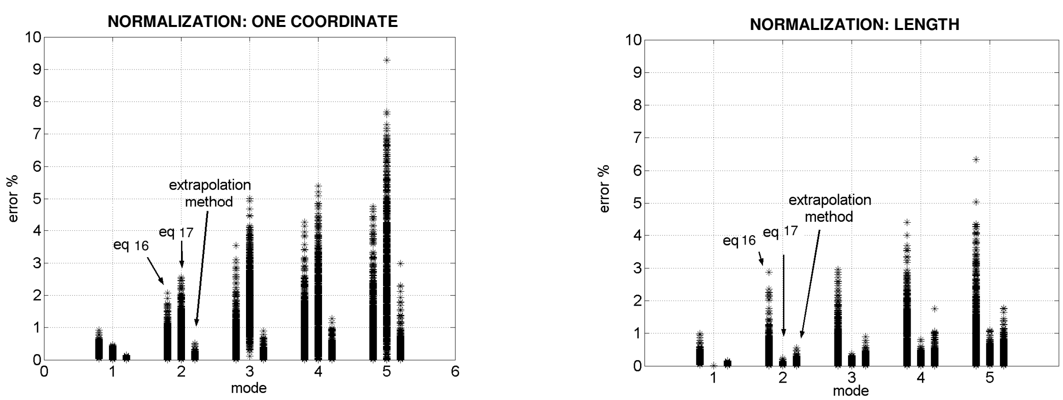

Fig. 4. Maximum errors in the first 5 modes using equations (16) and (17) and the extrapolation method. Mass change in 10 DOF's. Total mass change 10%,

Simulations have also been performed with other dynamic models and for different mass changes and the same conclusions can be extracted.

5.2 Accuracy of the methods.

In this paragraph, the accuracy obtained for equation (16), equation (17) , the extrapolation method and the coupled method are studied. Normalization to the length was used with all methods. In case of the extrapolation method, the magnitude of the first and of the second mass change are respectively $[\Delta m]$ and $2 \cdot [\Delta m]$.

One thousand simulations were performed on the simple dynamic system used in the former section. Systems with, 5, 6, 10, 15 and 20 degree of freedoms were used in the investigation and only the first 5 modes were considered.The mode shapes were normalized to the length and mass changes were performed placing masses in 3, 5, 10 and 15 random points.

The maximum errors in the first five modes are presented in Table 1 which involve the worst position of the masses for the scaling factor estimation of the first 5 modes. The maximum error always correspond to the 5th mode so that the error in the first 4 modes will be less. Furthermore, due to the fact that the scaling factors have to be calculated from modal parameters estimated in two stages, the best position of the masses can be selected from the unmodified mode shapes and the accuracy can be improved.

The accuracy improves as the number of observation points are reduced because in that case the masses are better distributed.

The best results are obtained using equation (17), except when the number of modes are close to the number of observation points, in which case the coupled method is better. When the number of modes is equal to the number of the observation points, the coupled method leads to exact results.

Equation (16) and the coupled method provides approximately the same accuracy.

The extrapolation method provides good results when the number of mass points are equal or larger than the number of modes. This method provides better results than Equation (16) and the coupled method.

Table 1. Maximum errors for the first five modes using the proposed mass change methods.

Number of		TOTAL Mass change	Mass change points	Maximum error (%) for the first 5 modes.			
DOF's	Modes			Equation (16) $\alpha=\sqrt{\frac{(\omega_0^2-\omega_1^2)}{\omega_1^2\cdot\{\psi_0\}^T\cdot[\Delta m]\cdot\{\psi_0\}}}$	Equation (17) $\alpha=\sqrt{\frac{(\omega_0^2-\omega_1^2)}{\omega_1^2\cdot\{\psi_0\}^T\cdot[\Delta m]\cdot\{\psi_1\}}}$	Coupled method	Extrapolation method
20	5	10%	3	28	5.5	30	22
			5	18	3.25	22	7.5
			10	5.5	1	8	3
			15	2	0.4	2.2	0.8
15	5	10%	3	25	5.2	26.5	21
			5	15	3	17	6
			10	4	0.4	7	1.2
10	5	10%	3	18	4	16	17
			5	12	1.65	12	5.5
6	5	10%	3	13	5	12	11
			5	2	0.75	0.65	0.55
5	5	10%	3	12	3	0	2.5

In Figure 5 the results of a 20 degree of freedom system with mass change in 5 and 10 points are shown. As can be seen, the coupled method is less accurate. In general, the accuracy with this method is of the same order that the extrapolation method in the first 75-80% of the modes, but in the rest of the modes the error increases significantly.

5. CONCLUSSIONS

- Two new methods are proposed to determine the scaling factors, when natural input modal analysis is performed.
- The accuracy obtained in the scaling factor estimation depends on the type of normalization used in the mode shapes. Two types of normalization are studied: normalization to unity and normalization to the length. Normalization to the length provides the best results.

- The best results are obtained using equation (17), except when the number of modes are close to the number of observation points, in which case the coupled method is better. When the number of modes is equal to the number of the observation points, the coupled method provides exact results.

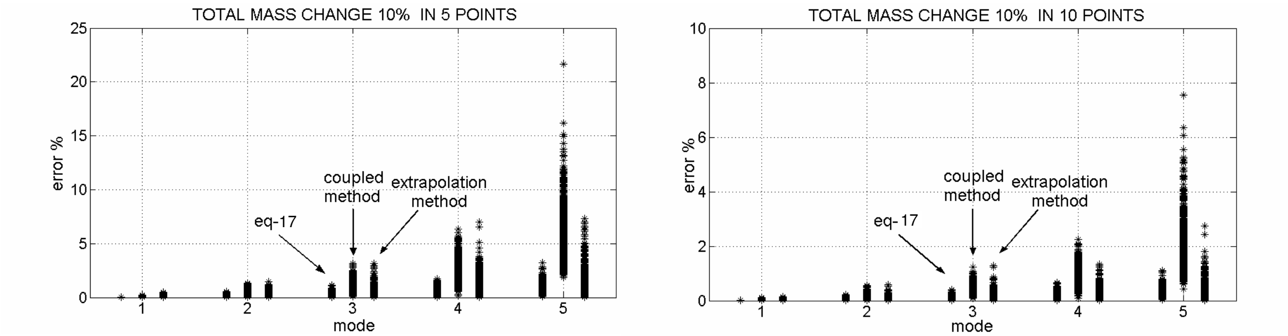

Fig. 5. Maximum errors in the first 5 modes. Mass change in 5 (left) and 10 (right) DOF's. Total mass change 10%.

6. ACKNOWLEDGEMENTS

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